MODULI OF PARABOLIC CONNECTIONS ON A CURVE AND RIEMANN-HILBERT CORRESPONDENCE

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ABSTRACT. Let $(C, \mathbf{t})$ ($\mathbf{t} = (t_1, \dots, t_n)$) be an n-pointed smooth projective curve of genus g and take $\boldsymbol{\lambda} = (\lambda_j^{(i)}) \in \mathbf{C}^{nr}$ such that $\sum_{i,j} \lambda_j^{(i)} = d \in \mathbf{Z}$. For a weight $\boldsymbol{\alpha}$, let $M_C^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda})$ be the moduli space of $\boldsymbol{\alpha}$-stable $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connections on C and for a certain $\mathbf{a} \in \mathbf{C}^{nr}$ let $RP_r(C, \mathbf{t})_{\mathbf{a}}$ be the moduli space of representations of the fundamental group $\pi_1(C \setminus \{t_1, \dots, t_n\}, *)$ with the local monodromy data $\mathbf{a}$. Then we prove that the morphism $\mathbf{RH} : M_C^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}) \to RP_r(C, \mathbf{t})_{\mathbf{a}}$ determined by Riemann-Hilbert correspondence is a proper surjective bimeromorphic morphism. As a corollary, we prove geometric Painlevé property of the isomonodromic deformation defined on the moduli space of parabolic connections.

1. Introduction

There are many kinds of description of Riemann-Hilbert correspondence. We consider here a moduli theoretic aspect of Riemann-Hilbert correspondence on a smooth projective curve. Even in the case of curve, moduli theoretic description of Riemann-Hilbert correspondence is in fact non-trivial and we will give a complete geometric description of Riemann-Hilbert correspondence from the view point of moduli theory. Moreover, we will introduce a good application of this description to the differential equation determined by isomonodromic deformation.

Let C be a smooth projective curve over $\mathbf{C}$ and $t_1, \dots, t_n$ be distinct points of C. For an algebraic vector bundle E on C and a logarithmic connection $\nabla : E \to E \otimes \Omega_C^1(t_1 + \cdots + t_n)$, $\ker \nabla^{an}|_{C \setminus \{t_1, \dots, t_n\}}$ becomes a local system on $C \setminus \{t_1, \dots, t_n\}$ and corresponds to a representation of the fundamental group $\pi_1(C \setminus \{t_1, \dots, t_n\}, *)$, where ∇^{an} is the analytic connection corresponding to ∇. The correspondence $(E, \nabla) \mapsto \ker \nabla^{an}|_{C \setminus \{t_1, \dots, t_n\}}$ is said to be Riemann-Hilbert correspondence. If $l \subset E|_{t_i}$ is a subspace satisfying $\mathsf{res}_{t_i}(\nabla)(l) \subset l$, then ∇ induces a connection $\nabla' : E' \to E' \otimes \Omega_C^1(t_1 + \cdots + t_n)$, where $E' := \ker(E \to (E|_{t_i}/l))$. We say (E', ∇') the elementary transform of (E, ∇) along t_i by l. Note that $\ker \nabla^{an}|_{C \setminus \{t_1, \dots, t_n\}} \cong \ker(\nabla')^{an}|_{C \setminus \{t_1, \dots, t_n\}}$. Roughly speaking, Riemann-Hilbert correspondence gives a bijection between the set

$$\left\{ (E, \nabla) \;\middle|\; \begin{array}{l} E \text{ is an algebraic vector bundle on } C \text{ of rank } r \text{ and} \\ \nabla : E \to E \otimes \Omega_C^1(t_1 + \cdots + t_n) \text{ is a logarithmic connection} \end{array} \right\} \Big/ {\scriptstyle \begin{array}{c} modulo \\ elementary\ transform \end{array}}$$

and the set

$$\{GL_r(\mathbf{C}) \text{ representation of } \pi_1(C \setminus \{t_1, \dots, t_n\}, *)\} \,/ \cong .$$

In order to define the former set exactly, we should introduce the concept of "parabolic structure". So put

$$\Lambda_r^{(n)}(d) := \left\{ (\lambda_j^{(i)})^{1 \le i \le n}_{0 \le j \le r-1} \in \mathbf{C}^{nr} \;\middle|\; d + \sum_{i,j} \lambda_j^{(i)} = 0 \right\}$$

for integers d, r, n with $r > 0$, $n > 0$. We write $\mathbf{t} = (t_1, \dots, t_n)$ and take $\boldsymbol{\lambda} \in \Lambda_r^{(n)}(d)$.

Definition 1.1. $(E, \nabla, \{l_*^{(i)}\}_{1 \le i \le n})$ is said to be a $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connection of rank r if

1. E is a rank r algebraic vector bundle on C,
2. $\nabla : E \to E \otimes \Omega_C^1(t_1 + \cdots + t_n)$ is a connection, and
3. for each t_i, $l_*^{(i)}$ is a filtration $E|_{t_i} = l_0^{(i)} \supset l_1^{(i)} \supset \cdots \supset l_{r-1}^{(i)} \supset l_r^{(i)} = 0$ such that $\dim(l_j^{(i)}/l_{j+1}^{(i)}) = 1$ and $(\mathsf{res}_{t_i}(\nabla) - \lambda_j^{(i)} \mathrm{id}_{E|_{t_i}})(l_j^{(i)}) \subset l_{j+1}^{(i)}$ for $j = 0, \dots, r-1$.

The filtraion $l_*^{(i)}$ $(1 \le i \le n)$ is said to be a parabolic structure of the vector bundle E. For a parabolic connection $(E, \nabla, \{l_j^{(i)}\})$, the elementary transform of (E, ∇) along t_i by $l_j^{(i)}$ determines another parabolic connection. Then an elementary transform gives a transformation $Elm_{l_j^{(i)}}$ on the set of isomorphism

classes of parabolic connections. Permutation of $(\lambda^{(i)}_0,\dots,\lambda^{(i)}_{r-1})$ induces a tranform a_i and tensoring $\mathcal{O}_C(t_i)$ induces a transform b_i. See (4), (3) and (5) in section 3 for the precise definition of $Elm_{l^{(i)}_j}$, a_i and b_i. Then we should precisely say that Riemann-Hilbert correspondence gives a bijection between the set

$$\left\{(E,\nabla,\{l^{(i)}_j\}) : \text{parabolic connection}\right\}\Big/\left\langle a_i, b_i, Elm_{l^{(i)}_j} \,\Big|\, 1\le i\le n, 0\le j\le r-1\right\rangle$$

and the set

$$\{\rho : \pi_1(C\setminus\{t_1,\dots,t_n\},*) \to GL_r(\mathbf{C}) : \text{representation}\} \,/\cong$$

We can easily see the proof of this bijection from the theory of Deligne ([4]).

We simply say the element of the former set a parabolic connection modulo elementary transform. Then a parabolic connection modulo elementary transform corresponds to a D module but we can not expect an appropriate algebraic structure on the moduli space of such objects. So we can recognize that it is natural to consider the moduli space of parabolic connections in the moduli theoretic description of Riemann-Hilbert correspondence. However, we must consider stability when we construct the moduli of parabolic connections as an appropriate space. So we set $M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda})$ as the moduli space of $\boldsymbol{\alpha}$-stable parabolic connections. See Theorem 2.1 and Definition 2.2 for the precise definition of $M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda})$. Next we consider the moduli space $RP_r(C,\mathbf{t})_{\mathbf{a}}$ of certain equivalence classes of representations of the fundamental group $\pi_1(C\setminus\{t_1,\dots,t_n\},*)$. Here two representations are equivalent if their semisimplifications are isomorphic. There is not an appropriate moduli space of isomorphism classes of the representations of the fundamental group. For a construction of a good moduli space containing all the representations, we must consider such an equivalence relation. See section 2.2 for the precise definition of $RP_r(C,\mathbf{t})_{\mathbf{a}}$. The most crucial part of the main result (Theorem 2.2) is that the morphism $\mathbf{RH} : M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda}) \longrightarrow RP_r(C,\mathbf{t})_{\mathbf{a}}$ determined by Riemann-Hilbert correspondece is a proper surjective bimeromorphic morphism. This theorem was proved in [7] for $C=\mathbf{P}^1$ and $r=2$. However there were certain difficulties to generalize this fact to the case of general C and r.

One of the most important motivation to consider such a theorem is to consider an application to the geometric description of the differential equation determined by isomonodromic deformation. See section 2 and section 7 for the precise definition of isomonodromic deformation. This differential equation is said to be Schlesinger equation for $C=\mathbf{P}^1$, Garnier equation for $C=\mathbf{P}^1$ and $r=2$, and Painlevé equation of type sixth for $C=\mathbf{P}^1$, $r=2$ and $n=4$. Jimbo, Miwa and Ueo give in [12] and [13] an explicit desecription of Schlesinger equation. However there was no description of the space where Schlesinger equation is defined which has the property that any analytic continuation of a solution of Schlesinger equation stays in the space. We call the property satisfied by such a space "geometric Painlevé property". See Definition 2.5 for the precise definition of geometric Painlevé property. Our aim here is to construct a space where isomonodromic deformation is defined and satisfies geometric Painlevé property over all values of $\boldsymbol{\lambda}$. (It is not difficult to construct such a space over generic $\boldsymbol{\lambda}$ but it is difficult to construct over special $\boldsymbol{\lambda}$.) In fact that space is nothing but the moduli space of $\boldsymbol{\alpha}$-stable parabolic connections and the result is given in Theorem 2.3, which is essentially a corollary of Theorem 2.2. Geometric Painlevé property immediately deduces usual analytic Painlevé property. So we can say that Theorem 2.3 gives a most clear proof of Painlevé property of isomonodromic deformation. As is well-known, the solutions of Painlevé equation have Painlevé property, which is in some sense the property characterizing Painlevé equation (and there were many proof of Painlevé property). So usual analytic Painlevé property plays an important role in the theory of Painlevé equation, but we can see from the definition that " geometric Painlevé property" is much more important from the view point of description of the geometric picture of isomonodromic deformation.

To prove the main results, this paper consists of several sections. In section 3, we prove the existence of the moduli space of stable parabolic connections. An algebraic moduli space of parabolic connections was essentially considered by D. Arinkin and S. Lysenco in [1], [2] and [3] and they showed that the moduli space of parabolic connections on $\mathbf{P}^1$ of rank 2 with $n=4$ for generic $\boldsymbol{\lambda}$ is isomorphic to the space of initial conditions of Painlevé equation of type sixth constructed by Okamoto ([15]). For special $\boldsymbol{\lambda}$, we should consider certain stability condition to construct an appropriate moduli space of parabolic connections. In the case of $C=\mathbf{P}^1$ and $r=2$, K. Iwasaki, M.-H. Saito and the author already considered in [7] the moduli space of stable parabolic connections and they proved in [8] that the moduli space of stable parabolic connections on $\mathbf{P}^1$ of rank 2 with $n=4$ is isomorphic to the space of initial conditions of Painlevé VI equation constructed by Okamoto for all $\boldsymbol{\lambda}$. An analytic construction of the moduli space

of stable parabolic connections for general C and $r = 2$ was given by H. Nakajima in [14]. However, in our aim, the algebraic construction of the moduli space is necessary. The morphism **RH** determined by Riemann-Hilbert correspondence is quite transcendental, which is explicitly shown in the case of $C = \mathbf{P}^1$, $r = 2$ and $n = 4$ in [8]. This statement make sense only if we construct the moduli spaces $M_C^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda})$ and $RP_r(C, \mathbf{t})_{\mathbf{a}}$ algebraically. For the proof of Theorem 2.1, we do not use the method in [16]. We construct $M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d)$ as a subscheme of the moduli space of parabolic Λ^1_D-triples constructed in [7]. We also use this embedding in the proof of the properness of **RH** in Theorem 2.2.

In section 4, we prove that the moduli space of stable parabolic connections is an irreducible variety. For the proof, we need cetain complicated calculations and this part is a new difficulty, which did not appear in [7].

In section 5, we consider the morphism **RH** determined by Riemann-Hilbert correspondence and prove the surjectivity and properness of **RH**. This is the essential part of the proof of Theorem 2.2. Notice that for generic $\boldsymbol{\lambda}$, parabolic connection is irreducible and so all parabolic connections are stable. Moreover we can easily see that the moduli space $M_C^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda})$ of parabolic connections for generic $\boldsymbol{\lambda}$ is analytically isomorphic via Riemann-Hilbert correspondence to the moduli space $RP_r(C, \mathbf{t})_{\mathbf{a}}$ of representations of the fundamental group $\pi_1(C \setminus \{t_1, \ldots, t_n\}, *)$. However, for special $\boldsymbol{\lambda}$, several parabolic connections may not be stable and the stability condition depends on $\boldsymbol{\alpha}$. So we can see that the surjectivity of **RH** is not trivial at all for specaial $\boldsymbol{\lambda}$, because for a point $[\rho] \in RP_r(C, \mathbf{t})_{\mathbf{a}}$, we must find a parabolic connection corresponding to $[\rho]$ which is $\boldsymbol{\alpha}$-*stable*. Moreover, the moduli space $M_C^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda})$ of stable parabolic connections is smooth, but the moduli space $RP_r(C, \mathbf{t})_{\mathbf{a}}$ of representations of the fundamental group becomes singular. So the morphism $\mathbf{RH} : M_C^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}) \to RP_r(C, \mathbf{t})_{\mathbf{a}}$ is more complicated in the case of special $\boldsymbol{\lambda}$ than the case of generic $\boldsymbol{\lambda}$. We first prove the surjectivity of **RH** in Proposition 5.1. In this proof, we use Langton's type theorem in the case of parabolic connections. This idea was already used in [7]. Secondly, we prove in Proposition 5.2 that every fiber of **RH** is compact by using an embedding of $M_C^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda})$ to a certain compact moduli space. In this proof, we can not use the idea given in [7] and make new technique again. Finally we obtain the properness of **RH** by the lemma given by A. Fujiki.

In section 6, we construct a canonical symplectic form on the moduli space $M_C^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda})$ of stable parabolic connections. Combined with the the fact that **RH** gives an analytic resolution of singularities of $RP_r(C, \mathbf{t})_{\mathbf{a}}$, we can say that $RP_r(C, \mathbf{t})_{\mathbf{a}}$ has symplectic singularities (for special $\mathbf{a}$) and **RH** gives a symplectic resolution of singularities. For the case of $r = 2$, H. Nakajima constructed the moduli space $M_C^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda})$ as a hyper-Kähler manifold and it obviously has a holomorphic symplectic structure. Such a construction for general r is also an important problem, though we do not treat it here.

In section 7, we first give in Proposition 7.1 an algebraic construction of the differential equation on $M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d)$ determined by isomonodromic deformation. Finally we complete the proof of Theorem 2.3.

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2. Main results

Let C be a smooth projective curve of genus g. We put

$$T_n := \left\{ (t_1, \ldots, t_n) \in \overbrace{C \times \cdots \times C}^{n} \,\middle|\, t_i \neq t_j \text{ for } i \neq j \right\}$$

for a positive integer n. For integers d, r with $r > 0$, we put

$$\Lambda_r^{(n)}(d) := \left\{ (\lambda_j^{(i)})^{1 \leq i \leq n}_{0 \leq j \leq r-1} \in \mathbf{C}^{nr} \,\middle|\, d + \sum_{i,j} \lambda_j^{(i)} = 0 \right\}.$$

Take a member $\mathbf{t} = (t_1, \ldots, t_n) \in T_n$ and $\boldsymbol{\lambda} = (\lambda_j^{(i)})_{1 \leq i \leq n, 0 \leq j \leq r-1} \in \Lambda_r^{(n)}(d)$.

Definition 2.1. $(E, \nabla, \{l_*^{(i)}\}_{1 \leq i \leq n})$ is said to be a $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connection of rank r if

(1) E is a rank r algebraic vector bundle on C,
(2) $\nabla : E \to E \otimes \Omega^1_C(t_1 + \cdots + t_n)$ is a connection, that is, ∇ is a homomorphism of sheaves satisfying $\nabla(fa) = a \otimes df + f\nabla(a)$ for $f \in \mathcal{O}_C$ and $a \in E$, and

(3) for each t_i, $l_*^{(i)}$ is a filtration $E|_{t_i} = l_0^{(i)} \supset l_1^{(i)} \supset \cdots \supset l_{r-1}^{(i)} \supset l_r^{(i)} = 0$ such that $\dim(l_j^{(i)}/l_{j+1}^{(i)}) = 1$ and $(\mathsf{res}_{t_i}(\nabla) - \lambda_j^{(i)} \mathrm{id}_{E|_{t_i}})(l_j^{(i)}) \subset l_{j+1}^{(i)}$ for $j = 0, \ldots, r-1$.

Remark 2.1. By condition (3) above, we have

$$\deg E = \deg(\det(E)) = -\sum_{i=1}^{n} \mathsf{res}_{t_i}(\nabla_{\det E}) = -\sum_{i=1}^{n}\sum_{j=0}^{r-1} \lambda_j^{(i)} = d.$$

Take rational numbers

$$0 < \alpha_1^{(i)} < \alpha_2^{(i)} < \cdots < \alpha_r^{(i)} < 1$$

for $i = 1, \ldots, n$ satisfying $\alpha_j^{(i)} \neq \alpha_{j'}^{(i')}$ for $(i,j) \neq (i',j')$. We choose $\boldsymbol{\alpha} = (\alpha_j^{(i)})$ sufficiently generic.

Definition 2.2. A parabolic connection $(E, \nabla, \{l_*^{(i)}\}_{1 \le i \le n})$ is $\boldsymbol{\alpha}$-stable (resp. $\boldsymbol{\alpha}$-semistable) if for any proper nonzero subbundle $F \subset E$ satisfying $\nabla(F) \subset F \otimes \Omega_C^1(t_1 + \cdots + t_n)$, the inequality

$$\frac{\deg F + \sum_{i=1}^n \sum_{j=1}^r \alpha_j^{(i)} \dim((F|_{t_i} \cap l_{j-1}^{(i)})/(F|_{t_i} \cap l_j^{(i)}))}{\operatorname{rank} F} \underset{(\text{resp. } \le)}{<} \frac{\deg E + \sum_{i=1}^n \sum_{j=1}^r \alpha_j^{(i)} \dim(l_{j-1}^{(i)}/l_j^{(i)})}{\operatorname{rank} E}$$

holds.

Let T be a smooth algebraic scheme which is a certain covering of the moduli stack of n-pointed smooth projective curves of genus g over $\mathbf{C}$ and take a universal family $(\mathcal{C}, \tilde{t}_1, \ldots, \tilde{t}_n)$ over T.

Theorem 2.1. *There exists a relative fine moduli scheme*

$$M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d) \to T \times \Lambda_r^{(n)}(d)$$

of $\boldsymbol{\alpha}$-stable parabolic connections of rank r and degree d, which is smooth and quasi-projective. The fiber $M^{\boldsymbol{\alpha}}_{\mathcal{C}_x}(\tilde{\mathbf{t}}_x, \boldsymbol{\lambda})$ over $(x, \boldsymbol{\lambda}) \in T \times \Lambda_r^{(n)}(d)$ is the moduli space of $\boldsymbol{\alpha}$-stable $(\tilde{\mathbf{t}}_x, \boldsymbol{\lambda})$ parabolic connections whose dimension is $2r^2(g-1) + nr(r-1) + 2$ if it is non-empty.

Definition 2.3. Take $\boldsymbol{\lambda} \in \Lambda_r^{(n)}(d)$. We call $\boldsymbol{\lambda}$ special if

(1) $\lambda_j^{(i)} - \lambda_k^{(i)} \in \mathbf{Z}$ for some i and $j \neq k$, or
(2) there exist an integer s with $1 < s < r$ and a subset $\{j_1^i, \ldots, j_s^i\} \subset \{0, \ldots, r-1\}$ for each $1 \le i \le n$ such that

$$\sum_{i=1}^{n}\sum_{k=1}^{s} \lambda_{j_k^i}^{(i)} \in \mathbf{Z}.$$

We call $\boldsymbol{\lambda}$ resonant if it satisfies the condition (1) above. If $\boldsymbol{\lambda}$ satisfies the condition (2) above, there is a reducible parabolic $(\tilde{\mathbf{t}}_x, \boldsymbol{\lambda})$-parabolic connection on $\mathcal{C}_x$ for $x \in T$. Here we say a $(\tilde{\mathbf{t}}_x, \boldsymbol{\lambda})$-connection $(E, \nabla_E, \{l_j^{(i)}\})$ reducible if there is a non-trivial subbundle $0 \neq F \subsetneq E$ such that $\nabla_E(F) \subset F \otimes \Omega^1_{\mathcal{C}_x}((\tilde{t}_1)_x + \cdots + (\tilde{t}_n)_x)$. We call $\boldsymbol{\lambda} \in \Lambda_r^{(n)}(d)$ generic if it is not special.

Fix a point $x \in T$. Then the fundamental group $\pi_1(\mathcal{C}_x \setminus \{(\tilde{t}_1)_x, \ldots, (\tilde{t}_n)_x\}, *)$ is generated by cycles $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ and loops γ_i around t_i for $1 \le i \le n$ whose relation is given by

$$\prod_{j=1}^{g} \alpha_j^{-1}\beta_j^{-1}\alpha_j\beta_j \prod_{i=1}^{n} \gamma_i = 1.$$

So the fundamental group is isomorphic to a free group generated by $2g + n - 1$ free generators. Then the space

$$\operatorname{Hom}\left(\pi_1(\mathcal{C}_x \setminus \{(\tilde{t}_1)_x, \ldots, (\tilde{t}_n)_x\}, *), GL_r(\mathbf{C})\right)$$

of representations of the fundamental group becomes an affine variety isomorphic to $GL_r(\mathbf{C})^{2g+n-1}$ and $GL_r(\mathbf{C})$ acts on this space by the adjoint action. We define

$$RP_r((\mathcal{C})_x, \tilde{\mathbf{t}}_x) = \operatorname{Hom}\left(\pi_1(\mathcal{C}_x \setminus \{(\tilde{t}_1)_x, \ldots, (\tilde{t}_n)_x\}, *), GL_r(\mathbf{C})\right) // GL_r(\mathbf{C})$$

as a categorical quotient. If we put

$$\mathcal{A}_r^{(n)} := \left\{ \mathbf{a} = (a_j^{(i)})^{1\leq i\leq n}_{0\leq j\leq r-1} \in \mathbf{C}^{nr} \,\middle|\, a_0^{(1)} a_0^{(2)} \cdots a_0^{(n)} = (-1)^{rn} \right\},$$

then we can define a morphism

$$RP_r((\mathcal{C})_x, \tilde{\mathbf{t}}_x) \longrightarrow \mathcal{A}_r^{(n)}$$
$$[\rho] \mapsto \mathbf{a} = (a_j^{(i)})$$

by the relation

$$\det(XI_r - \rho(\gamma_i)) = X^r + a_{r-1}^{(i)} X^{r-1} + \cdots + a_0^{(i)},$$

where X is an indeterminate and I_r is the identity matrix of size r. Note that for any $GL_r(\mathbf{C})$-representation ρ of $\pi_1(\mathcal{C}_x \setminus \{(\tilde{t}_1)_x, \ldots, (\tilde{t}_n)_x\}, *)$,

$$\begin{aligned}
&\det(\rho(\gamma_1)) \cdots \det(\rho(\gamma_n)) \\
&= \det(\rho(\beta_g))^{-1} \det(\rho(\alpha_g))^{-1} \det(\beta_g) \det(\alpha_g) \cdots \det(\rho(\beta_1))^{-1} \det(\rho(\alpha_1))^{-1} \det(\beta_1) \det(\alpha_1) \\
&= 1
\end{aligned}$$

and so the equation $a_0^{(1)} a_0^{(2)} \cdots a_0^{(n)} = (-1)^{rn}$ should be satisfied.

Replacing T by a covering, we can define a relative moduli space $RP_r(\mathcal{C}, \tilde{\mathbf{t}}) = \coprod_{x\in T} RP_r(\mathcal{C}_x, \tilde{\mathbf{t}}_x)$ of representations with a morphism

$$RP_r(\mathcal{C}, \tilde{\mathbf{t}}) \longrightarrow T \times \mathcal{A}_r^{(n)}.$$

We define a morphism

$$rh : \Lambda_r^{(n)}(d) \ni \boldsymbol{\lambda} \mapsto \mathbf{a} \in \mathcal{A}_r^{(n)} \tag{1}$$

by

$$\prod_{j=0}^{r-1} \left(X - \exp(-2\pi\sqrt{-1}\lambda_j^{(i)}) \right) = X^r + a_{r-1}^{(i)} X^{r-1} + \cdots + a_0^{(i)}.$$

For each member $(E, \nabla, \{l_j^{(i)}\}) \in M^{\boldsymbol{\alpha}}_{\mathcal{C}_x}(\tilde{\mathbf{t}}_x, \boldsymbol{\lambda})$, $\ker(\nabla^{an}|_{\mathcal{C}_x \setminus \{(\tilde{t}_1)_x, \ldots, (\tilde{t}_n)_x\}})$ becomes a local system on $\mathcal{C}_x \setminus \{(\tilde{t}_1)_x, \ldots, (\tilde{t}_n)_x\}$, where ∇^{an} means the analytic connection corresponding to ∇. The local system $\ker(\nabla^{an}|_{\mathcal{C}_x \setminus \{(\tilde{t}_1)_x, \ldots, (\tilde{t}_n)_x\}})$ corresponds to a representation of $\pi_1(\mathcal{C}_x \setminus \{(\tilde{t}_1)_x, \ldots, (\tilde{t}_n)_x\}, *)$. So we can define a morphism

$$\mathbf{RH}_{(x,\boldsymbol{\lambda})} : M^{\boldsymbol{\alpha}}_{\mathcal{C}_x}(\tilde{\mathbf{t}}_x, \boldsymbol{\lambda}) \longrightarrow RP_r(\mathcal{C}_x, \tilde{\mathbf{t}}_x)_{\mathbf{a}},$$

where $\mathbf{a} = rh(\boldsymbol{\lambda})$. Replacing T by a covering again, $\{\mathbf{RH}_{(x,\boldsymbol{\lambda})}\}$ induces a morphism

$$\mathbf{RH} : M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d) \longrightarrow RP_r(\mathcal{C}, \tilde{\mathbf{t}}) \tag{2}$$

which makes the diagram

$$\begin{array}{ccc}
M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d)) & \xrightarrow{\mathbf{RH}} & RP_r(\mathcal{C}, \tilde{\mathbf{t}}) \\
\big\downarrow & & \big\downarrow \\
T \times \Lambda_r^{(n)}(d) & \xrightarrow{\mathrm{id}\times rh} & T \times \mathcal{A}_r^{(n)}
\end{array}$$

commute.

The follwing is the main theorem whose proof is given in section 5.

Theorem 2.2. *Assume $\boldsymbol{\alpha}$ is so generic that $\boldsymbol{\alpha}$-stable $\Leftrightarrow$ $\boldsymbol{\alpha}$-semistable. Moreover we assume $rn-2r-2>0$ if $g=0$, $n>1$ if $g=1$ and $n\geq 1$ if $g\geq 2$. Then the morphism*

$$\mathbf{RH} : M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d) \longrightarrow RP_r(\mathcal{C}, \tilde{\mathbf{t}}) \times_{\mathcal{A}_r^{(n)}} \Lambda_r^{(n)}(d)$$

induced by (2) is a proper surjective bimeromorphic morphism. Combined with Proposition 6.1, we can say that for each $(x, \boldsymbol{\lambda}) \in T \times \Lambda_r^{(n)}(d)$,

1. $\mathbf{RH}_{(x,\boldsymbol{\lambda})} : M^{\boldsymbol{\alpha}}_{\mathcal{C}_x}(\tilde{\mathbf{t}}_x, \boldsymbol{\lambda}) \longrightarrow RP_r(\mathcal{C}_x, \tilde{\mathbf{t}}_x)_{\mathbf{a}}$ *is an analytic isomorphisms of symplectic varieties for generic $\boldsymbol{\lambda}$ and*
2. $\mathbf{RH}_{(x,\boldsymbol{\lambda})} : M^{\boldsymbol{\alpha}}_{\mathcal{C}_x}(\tilde{\mathbf{t}}_x, \boldsymbol{\lambda}) \longrightarrow RP_r(\mathcal{C}_x, \tilde{\mathbf{t}}_x)_{\mathbf{a}}$ *gives an analytic symplectic resolution of singularities of $RP_r(\mathcal{C}_x, \tilde{\mathbf{t}}_x)_{\mathbf{a}}$ for special $\boldsymbol{\lambda}$.*

Here we put $\mathbf{a} = rh(\boldsymbol{\lambda})$.

Remark 2.2. (1) For the case of $r=1$, $(\mathbf{RH})_{(x,\boldsymbol{\lambda})}$ is an isomorphism for any $(x,\boldsymbol{\lambda}) \in T \times \Lambda_r^{(n)}(d)$ and $RP_1(\mathcal{C}_x, \tilde{\mathbf{t}}_x)_{\mathbf{a}}$ is smooth for any $\mathbf{a}$.
(2) In Theorem 2.2 we consider the case $n \geq 1$. As is stated in [[17], Proposition 7.8], Riemann-Hilbert correspondence for the case of $n=0$ gives an analytic isomorphism between the moduli space of integrable connections and the moduli space of representations of the fundamental group, but in that case the moduli space of integrable connections may be singular.

Let $\tilde{T} \to T$ be the universal covering. Then $RP_r(\mathcal{C}, \tilde{\mathbf{t}}) \times_T \tilde{T} \to \tilde{T}$ becomes a trivial fibration and we can consider the set of constant sections

$$\mathcal{F}_R := \left\{ \sigma : \tilde{T} \to RP_r(\mathcal{C}, \tilde{\mathbf{t}}) \times_T \tilde{T} \right\}.$$

As is stated in Remark 5.2,

$$RP_r(C, \mathbf{t})_{\mathbf{a}}^{\mathrm{sing}} = \left\{ [\rho] \in RP_r(C, \mathbf{t})_{\mathbf{a}} \;\middle|\; \begin{array}{l} \rho \text{ is reducible or} \\ \dim \left(\ker \left(\rho(\gamma_i) - \exp(-2\pi\sqrt{-1}\lambda_j^{(i)}) I_r \right) \right) \geq 2 \text{ for some } i, j \end{array} \right\}$$

is just the singular locus of $RP_r(C, \mathbf{t})_{\mathbf{a}}$. If we put $RP_r(\mathcal{C}_x, \tilde{\mathbf{t}}_x)^{\sharp}_{\mathbf{a}} := RP_r(\mathcal{C}_x, \tilde{\mathbf{t}}_x)_{\mathbf{a}} \setminus RP_r(C, \mathbf{t})_{\mathbf{a}}^{\mathrm{sing}}$, then $\mathbf{RH}|_{M^{\boldsymbol{\alpha}}_{\mathcal{C}_x}(\tilde{\mathbf{t}}_x, \boldsymbol{\lambda})^{\sharp}} : M^{\boldsymbol{\alpha}}_{\mathcal{C}_x}(\tilde{\mathbf{t}}_x, \boldsymbol{\lambda})^{\sharp} \to RP_r(\mathcal{C}_x, \tilde{\mathbf{t}}_x)^{\sharp}_{\mathbf{a}}$ becomes an isomorphism, where we put $M^{\boldsymbol{\alpha}}_{\mathcal{C}_x}(\tilde{\mathbf{t}}_x, \boldsymbol{\lambda})^{\sharp} := \mathbf{RH}^{-1}_{(x,\boldsymbol{\lambda})}(RP_r(\mathcal{C}_x, \tilde{\mathbf{t}}_x)^{\sharp}_{\mathbf{a}})$. Put

$$RP_r(\mathcal{C}, \tilde{\mathbf{t}})^{\sharp} = \coprod_{(x,\mathbf{a}) \in T \times \mathcal{A}_r^{(n)}} RP_r(\mathcal{C}_x, \tilde{\mathbf{t}}_x)^{\sharp}_{\mathbf{a}}.$$

Then the restriction $\mathcal{F}_R^{\sharp}$ of $\mathcal{F}_R$ to $RP_r(\mathcal{C}, \tilde{\mathbf{t}})^{\sharp} \times_T \tilde{T}$ gives a foliation on $RP_r(\mathcal{C}, \tilde{\mathbf{t}})^{\sharp} \times_T \tilde{T}$. The pull back $\tilde{\mathcal{F}}_M^{\sharp} := \mathbf{RH}^{-1}(\mathcal{F}_R^{\sharp})$ determines a foliation on $M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d)^{\sharp} \times_T \tilde{T}$ where $M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d)^{\sharp} = \mathbf{RH}^{-1}(RP_r(\mathcal{C}, \tilde{\mathbf{t}})^{\sharp})$. This foliation corresponds to a subbundle of the analytic tangent bundle $\Theta^{an}_{M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d)^{\sharp} \times_T \tilde{T}}$ determined by a splitting

$$D^{\sharp} : \tilde{\pi}^*(\Theta^{an}_{\tilde{T}}) \to \Theta^{an}_{M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d)^{\sharp} \times_T \tilde{T}}$$

of the analytic tangent map

$$\Theta^{an}_{M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d)^{\sharp} \times_T \tilde{T}} \longrightarrow \tilde{\pi}^* \Theta^{an}_{\tilde{T}} \to 0,$$

where $\tilde{\pi} : M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d)^{\sharp} \times_T \tilde{T} \to \tilde{T}$ is the projection. We will show in Proposition 7.1 that this splitting $D^{\sharp}$ is in fact induced by a splitting

$$D : \pi^*(\Theta_T) \to \Theta_{M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d)}$$

of the algebraic tangent bundle, where $\pi : M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d) \to T$ is the projection.

In order to explain the concept of geometric Painlevé property, let us review here several terminologies introduced by K. Iwasaki in [9].

Definition 2.4. ([9]) *A time-dependent dynamical system $(M, \mathcal{F})$ is a smooth fibration $\pi : M \to T$ of complex manifolds together with a complex foliation $\mathcal{F}$ on M that is transverse to each fiber $M_t = \pi^{-1}(t)$, $t \in T$. The total space M is referred to as the phase space, while the base space T is called the space of time-variables. Moreover, the fiber M_t is called the space of initial conditions at time t.*

Definition 2.5. ([9]) *A time-dependent dynamical system $(M, \mathcal{F})$ is said to have geometric Painlevé property if for any path γ in T and any point $p \in M_t$, where t is the initial point of γ, there exists a unique $\mathcal{F}$-horizontal lift $\tilde{\gamma}_p$ of γ with initial point p. Here a curve in M is said to be $\mathcal{F}$-horizontal if it lies in a leaf of $\mathcal{F}$.*

$D^{\sharp}(\tilde{\pi}^* \Theta^{an}_{\tilde{T}})$ obviously satisfies the integrability condition

$$[D^{\sharp}(\tilde{\pi}^* \Theta^{an}_{\tilde{T}}), D^{\sharp}(\tilde{\pi}^* \Theta^{an}_{\tilde{T}})] \subset D^{\sharp}(\tilde{\pi}^* \Theta^{an}_{\tilde{T}})$$

because it corresponds to the foliation $\tilde{\mathcal{F}}_M^{\sharp}$. Since $\mathrm{codim}_{M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d)} \left(M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d) \setminus M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d)^{\sharp} \right) \geq 2$, we can see that $D(\pi^*(\Theta_T))$ also satisfies the integrability condition

$$[D(\pi^*(\Theta_T)), D(\pi^*(\Theta_T))] \subset D(\pi^*(\Theta_T)).$$

Then the foliation $\tilde{\mathcal{F}}^{\sharp}_M$ extends to a foliation $\tilde{\mathcal{F}}_M$ on $M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d) \times_T \tilde{T}$. We can see that $\tilde{\mathcal{F}}_M$ descends to a foliation $\mathcal{F}_M$ on $M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d)$ which corresponds to the subbundle $D(\pi^*(\Theta_T)) \subset \Theta_{M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d)}$. By construction, we can see that $\left(M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d), \mathcal{F}_M\right)$ becomes a time-dependent dynamical system with base space T. As a corollary of Theorem 2.2, we can prove the following theorem whose proof is given in section 7.

Theorem 2.3. *The time-dependent dynamical system $\left(M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d), \mathcal{F}_M\right)$ determined by the differential system $D(\pi^*\Theta_T)$ satisfies geometric Painlevé property.*

3. Moduli of stable parabolic connections

Before proving Theorem 2.1, we recall the definition of parabolic Λ^1_D-triple defined in [7]. Let D be an effective divisor on a curve C. We define Λ^1_D as $\mathcal{O}_C \oplus \Omega^1_C(D)^\vee$ with the bimodule structure given by

$$\begin{aligned} f(a,v) &= (fa, fv) \quad (f, a \in \mathcal{O}_C, v \in \Omega^1_C(D)^\vee) \\ (a,v)f &= (fa + v(f), fv) \quad (f, a \in \mathcal{O}_C, v \in \Omega^1_C(D)^\vee). \end{aligned}$$

Definition 3.1. $(E_1, E_2, \Phi, F_*(E_1))$ is said to be a parabolic Λ^1_D-triple on C of rank r and degree d if

1. E_1 and E_2 are vector bundles on C of rank r and degree d,
2. $\Phi : \Lambda^1_D \otimes E_1 \to E_2$ is a left $\mathcal{O}_C$-homomorphism,
3. $E_1 = F_1(E_1) \supset F_2(E_1) \supset \cdots \supset F_l(E_1) \supset F_{l+1}(E_1) = E_1(-D)$ is a filtration by coherent subsheaves.

We take positive integers β_1, β_2, γ and rational numbers $0 < \alpha'_1 < \cdots < \alpha'_l < 1$. We assume $\gamma \gg 0$.

Definition 3.2. A parabolic Λ^1_D-triple $(E_1, E_2, \Phi, F_*(E_1))$ is $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma)$-stable (resp. $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma)$-semistable) if for any subbundles $(F_1, F_2) \subset (E_1, E_2)$ satisfying $(0,0) \neq (F_1, F_2) \neq (E_1, E_2)$ and $\Phi(\Lambda^1_{D(\mathbf{t})} \otimes F_1) \subset F_2$, the inequality

$$\frac{\beta_1 \deg F_1(-D) + \beta_2(\deg F_2 - \gamma \operatorname{rank} F_2) + \beta_1 \sum_{j=1}^{l} \alpha'_j \operatorname{length}((F_j(E_1) \cap F_1)/(F_{j+1}(E_1) \cap F_1))}{\beta_1 \operatorname{rank} F_1 + \beta_2 \operatorname{rank} F_2}$$

$$\underset{(\text{resp. } \leq)}{<} \frac{\beta_1 \deg E_1(-D) + \beta_2(\deg E_2 - \gamma \operatorname{rank} E_2) + \beta_1 \sum_{j=1}^{l} \alpha'_j \operatorname{length}(F_j(E_1)/F_{j+1}(E_1))}{\beta_1 \operatorname{rank} E_1 + \beta_2 \operatorname{rank} E_2}$$

holds.

Theorem 3.1. **([7], Theorem 7.1)** *Let S be a noetherian scheme, $\mathcal{C}$ be a flat family of smooth projective curves of genus g and D be an effective Cartier divisor on $\mathcal{C}$ flat over S. Then there exists a coarse moduli scheme $\overline{M^{D,\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}_{\mathcal{C}/S}}(r, d, \{1\}_{1 \leq i \leq nr})$ of $(\boldsymbol{\alpha}', \boldsymbol{\beta}, \gamma)$-stable parabolic Λ^1_D-triples on $\mathcal{C}$ over S. If $\boldsymbol{\alpha}'$ is generic, it is projective over S.*

Definition 3.3. We denote the pull-back of $\mathcal{C}$ and $\tilde{\mathbf{t}}$ by the morphism $T \times \Lambda^{(n)}_r(d) \to T$ by the same characters $\mathcal{C}$ and $\tilde{\mathbf{t}} = (\tilde{t}_1, \dots, \tilde{t}_n)$. Then $D(\tilde{\mathbf{t}}) := \tilde{t}_1 + \cdots + \tilde{t}_n$ becomes a Cartier divisor on $\mathcal{C}$ flat over $T \times \Lambda^{(n)}_r(d)$. We also denote by $\tilde{\boldsymbol{\lambda}}$ the pull-back of the univerasal family on $\Lambda^{(n)}_r(d)$ by the morphism $T \times \Lambda^{(n)}_r(d) \to \Lambda^{(n)}_r(d)$. We define a functor $\mathcal{M}^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d)$ of the category of locally noetherian schemes to the category of sets by

$$\mathcal{M}^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d)(S) := \left\{(E, \nabla, \{l^{(i)}_j\})\right\} / \sim,$$

for a locally noetherian scheme S over $T \times \Lambda^{(n)}_r(d)$, where

1. E is a vector bundle on $\mathcal{C}_S$ of rank r,
2. $\nabla : E \to E \otimes \Omega^1_{\mathcal{C}_S/S}(D(\tilde{\mathbf{t}})_S)$ is a relative connection,
3. $E|_{(\tilde{t}_i)_S} = l^{(i)}_0 \supset l^{(i)}_1 \supset \cdots \supset l^{(i)}_{r-1} \supset l^{(i)}_r = 0$ is a filtration by subbundles such that $(\mathrm{res}_{(\tilde{t}_i)_S}(\nabla) - (\tilde{\lambda}^{(i)}_j)_S)(l^{(i)}_j) \subset l^{(i)}_{j+1}$ for $0 \leq j \leq r-1$, $i = 1, \dots, n$,
4. for any geometric point $s \in S$, $\dim(l^{(i)}_j / l^{(i)}_{j+1}) \otimes k(s) = 1$ for any i, j and $(E, \nabla, \{l^{(i)}_j\}) \otimes k(s)$ is $\boldsymbol{\alpha}$-stable.

Here $(E,\nabla,\{l^{(i)}_j\})\sim(E',\nabla',\{l'^{(i)}_j\})$ if there exist a line bundle $\mathcal{L}$ on S and an isomorphism $\sigma : E\xrightarrow{\sim}E'\otimes\mathcal{L}$ such that $\sigma|_{t_i}(l^{(i)}_j)=l'^{(i)}_j$ for any i,j and the diagram

$$\begin{array}{ccc} E & \xrightarrow{\nabla} & E\otimes\Omega^1_{\mathcal{C}/T}(D(\tilde{\mathbf{t}})) \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma\otimes\mathrm{id}} \\ E'\otimes\mathcal{L} & \xrightarrow{\nabla'} & E'\otimes\Omega^1_{\mathcal{C}/T}(D(\tilde{\mathbf{t}}))\otimes\mathcal{L} \end{array}$$

commutes.

Proof of Theorem 2.1. Fix a weight $\boldsymbol{\alpha}$ which determines the stability of parabolic connections. We take positive integers β_1,β_2,γ and rational numbers $0<\tilde{\alpha}^{(i)}_1<\tilde{\alpha}^{(i)}_2<\cdots<\tilde{\alpha}^{(i)}_r<1$ satisfying $(\beta_1+\beta_2)\alpha^{(i)}_j=\beta_1\tilde{\alpha}^{(i)}_j$ for any i,j. We assume $\gamma\gg 0$. We can take an increasing sequence $0<\alpha'_1<\cdots<\alpha'_{nr}<1$ such that $\{\alpha'_i|1\le i\le nr\}=\left\{\tilde{\alpha}^{(i)}_j\,\middle|\,1\le i\le n,1\le j\le r\right\}$. Take any member $(E,\nabla,\{l^{(i)}_j\})\in\mathcal{M}^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)(T)$. For each $1\le p\le rn$, exist i,j satisfying $\tilde{\alpha}^{(i)}_j=\alpha'_p$. We define inductively $F_p(E):=\ker(F_{p-1}(E)\to(E)|_{t_i}/l^{(i)}_j)$. Then $(E,\nabla,\{l^{(i)}_j\})\mapsto(E,E,\phi,\nabla,F_*(E))$ determines a morphism

$$\iota:\mathcal{M}^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)\to\overline{\mathcal{M}^{D(\tilde{\mathbf{t}}),\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}_{\mathcal{C}/T\times\Lambda^{(n)}_r(d)}}(r,d,\{1\}_{1\le i\le nr}),$$

where $\overline{\mathcal{M}^{D(\tilde{\mathbf{t}}),\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}_{\mathcal{C}/T}}(r,d,\{1\}_{1\le i\le nr})$ is the moduli functor of $(\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma)$-stable parabolic $\Lambda^1_{D(\tilde{\mathbf{t}})}$-triples whose coase moduli scheme exists by Theorem 3.1. Note that a parabolic connection $(E,\nabla,\{l^{(i)}_j\})$ is $\boldsymbol{\alpha}$-stable if and only if the corresponding parabolic $\Lambda^1_{D(\mathbf{t})}$-triple $(E,E,\mathrm{id},\nabla,F_*(E))$ is $(\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma)$-stable since $\gamma\gg 0$. We can see that ι is representable by an immersion. So we can prove in the same way as [[7] Theorem 2.1] that a certain subscheme $M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)$ of $\overline{M^{D(\mathbf{t}),\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}_{C}}(r,d,\{1\}_{1\le i\le nr})$ is just the coarse moduli scheme of $\mathcal{M}^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)$.

Applying a certain elementary transformation, we obtain an isomorphism

$$\sigma:\mathcal{M}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)\xrightarrow{\sim}\mathcal{M}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d')$$

of moduli stacks of parabolic connections without stability condition, where d' and r are coprime. Then $\sigma(M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d))$ can be considered as the moduli scheme of parabolic connections of rank r and degree d' satisfying a certain stability condition. We can take a vector space V such that there exists a surjection $V\otimes\mathcal{O}_{\mathcal{C}_s}(-m)\to E$ such that $V\otimes k(s)\to H^0(E(m))$ is isomorphic and $h^i(E(m))=0$ for $i>0$ for any member $(E,\nabla,\{l^{(i)}_j\})\in\sigma(M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d))$ over $s\in T$. We put $P(n):=\chi(E(n))$. Let $\mathcal{E}$ be the universal family on $\mathcal{C}\times_{T\times\Lambda^{(n)}_r(d)}\mathrm{Quot}^P_{\mathcal{C}/T\times\Lambda^{(n)}_r(d)}$. Then we can construct a scheme R over $\mathrm{Quot}^P_{\mathcal{C}/T\times\Lambda^{(n)}_r(d)}$ which parametrizes connections $\nabla:\mathcal{E}_s\to\mathcal{E}_s\otimes\Omega^1_{\mathcal{C}_s}(D(\tilde{\mathbf{t}}_s))$ and parabolic structures $\mathcal{E}_s|_{(\tilde{t}_i)_s}=l^{(i)}_0\supset\cdots\supset l^{(i)}_{r-1}\supset l^{(i)}_r=0$ such that $(\mathsf{res}_{(\tilde{t}_i)_s}(\nabla)-(\tilde{\lambda}^{(i)}_j)_s)(l^{(i)}_j)\subset l^{(i)}_{j+1}$ for any i,j. $PGL(V)$ canonically acts on R and for some open subscheme R^s of R, there is a canonical morphism $R^s\to\sigma(M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d))$ which becomes a principal $PGL(V)$-bundle. Then we can prove in the same manner as [[6], Theorem 4.6.5] that for a certain line bundle $\mathcal{L}$ on R^s, $\mathcal{E}\otimes\mathcal{L}$ descends to a vector bundle on $\mathcal{C}\times_{T\times\Lambda^{(n)}_r(d)}\sigma(M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d))$, since r and d' are coprime. We can easily see that the universal families $\tilde{\nabla}$, $\{\tilde{l}^{(i)}_j\}$ of connections and parabolic structures on $\mathcal{E}\otimes\mathcal{L}$ also descends. So we obtain a universal family for the moduli space $\sigma(M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d))$ and $M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)$ in fact becomes a fine moduli scheme.

Let $(\tilde{E},\tilde{\nabla},\{\tilde{l}^{(i)}_j\})$ be a universal family on $\mathcal{C}\times_T M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)$. We define a complex $\mathcal{F}^\bullet$ by

$$\mathcal{F}^0:=\left\{s\in\mathcal{E}nd(\tilde{E})\,\middle|\,s|_{\tilde{t}_i\times M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)}(\tilde{l}^{(i)}_j)\subset\tilde{l}^{(i)}_j\text{ for any }i,j\right\}$$

$$\mathcal{F}^1:=\left\{s\in\mathcal{E}nd(\tilde{E})\otimes\Omega^1_{\mathcal{C}/T}(D(\tilde{\mathbf{t}}))\,\middle|\,\mathsf{res}_{\tilde{t}_i\times M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)}(s)(\tilde{l}^{(i)}_j)\subset\tilde{l}^{(i)}_{j+1}\text{ for any }i,j\right\}$$

$$\nabla_{\mathcal{F}^\bullet}:\mathcal{F}^0\longrightarrow\mathcal{F}^1;\quad\nabla_{\mathcal{F}^\bullet}(s)=\tilde{\nabla}\circ s-s\circ\tilde{\nabla}.$$

Let (A,m) be an artinian local ring over $T\times\Lambda_r^{(n)}(d)$, I an ideal of A satisfying $mI=0$. Take any member $(E,\nabla,\{l_j^{(i)}\})\in M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)(A/I)$. We denote the image of $\operatorname{Spec}A/m\to M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)$ by x. Then we can see that the obstruction class $\omega(E,\nabla,\{l_j^{(i)}\})$ for lifting $(E,\nabla,\{l_j^{(i)}\})$ to an A-valued point of $M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)$ lies in the cohomology $\mathbf{H}^2(\mathcal{F}^\bullet\otimes k(x))\otimes I$. Consider the homomorphism

$$\begin{aligned}\mathrm{Tr}:\mathbf{H}^2(\mathcal{F}^\bullet\otimes k(x))\otimes I&\longrightarrow\mathbf{H}^2(\Omega^\bullet_{\mathcal{C}_x})\otimes I\\ [\{u_{\alpha\beta\gamma}\},\{v_{\alpha\beta}\}]&\mapsto[\{\mathrm{Tr}(u_{\alpha\beta\gamma})\},\{\mathrm{Tr}(v_{\alpha\beta})\}],\end{aligned}$$

where $\mathcal{C}_x=\bigcup U_\alpha$ is an affine open covering and we consider in Čech cohomology. Then we can see that $\mathrm{Tr}(\omega(E,\nabla,\{l_j^{(i)}\}))$ is just the obstruction class for lifting $(\det E,\wedge^r\nabla)$ to a pair of a line bundle and a connection with the fixed residue $\det(\tilde{\boldsymbol{\lambda}}_A)$ over A. Here $\det(\tilde{\boldsymbol{\lambda}}_A)=(\sum_{j=0}^{r-1}(\tilde{\lambda}_j^{(i)})_A)^{1\leq i\leq n}$. Since $\operatorname{Pic}^d\mathcal{C}/T$ is smooth over T, there is a line bundle L on $\mathcal{C}_A$ such that $L\otimes A/I\cong\det(E)$. We can construct a connection $\tilde{\nabla}^0:L\to L\otimes\Omega^1_{\mathcal{C}/T}(\tilde{t}_1+\cdots+\tilde{t}_n)$ because $H^1(\mathcal{E}nd(L)\otimes\Omega^1_{\mathcal{C}/T}(\tilde{t}_1+\cdots+\tilde{t}_n))=0$. We can find a member $u\in H^0(\mathcal{E}nd(L)\otimes\Omega^1_{\mathcal{C}/T}(\tilde{t}_1+\cdots+\tilde{t}_n))$ such that $u\otimes A/I=\tilde{\nabla}^0\otimes A/I-\wedge^r\nabla$. Then $(L,\tilde{\nabla}^0-u)$ becomes a lift of $(\det(E),\wedge^r\nabla)$. If we put $\mu^{(i)}:=\mathsf{res}_{\tilde{t}_i\otimes A}(\tilde{\nabla}^0-u)-\sum_j(\tilde{\lambda}_j^{(i)})_A$, then we have $\mu_i\in I$ for any i and $\sum_{i=1}^n\mu^{(i)}=0$. We can find an element $\omega\in I\otimes H^0(\mathcal{E}nd(L)\otimes\Omega^1_{\mathcal{C}/T}(\tilde{t}_1+\cdots+\tilde{t}_n))$ such that $\mathsf{res}_{\tilde{t}_i\otimes A}(\omega)=\mu^{(i)}$ for any i. Then $(L,\tilde{\nabla}^0-u-\omega)$ is a lift of $(\det(E),\wedge^r\nabla)$ with the residue $(\det(\tilde{\boldsymbol{\lambda}})_A)$. Thus we have $\mathrm{Tr}(\omega(E,\nabla,\{l_j^{(i)}\}))=0$. Since $(\mathcal{F}^1)^\vee\otimes\Omega^1_{\mathcal{C}/T}\cong\mathcal{F}^0$ and $(\mathcal{F}^0)^\vee\otimes\Omega^1_{\mathcal{C}/T}\cong\mathcal{F}^1$, we have

$$\begin{aligned}\mathbf{H}^2(\mathcal{F}^\bullet\otimes k(x))&\cong\operatorname{coker}\left(H^1(\mathcal{F}^0\otimes k(x))\to H^1(\mathcal{F}^1\otimes k(x))\right)\\&\cong\ker\Big(H^0((\mathcal{F}^1)^\vee\otimes\Omega^1_{\mathcal{C}/T}\otimes k(x))\to H^0((\mathcal{F}^0)^\vee\otimes\Omega^1_{\mathcal{C}/T}\otimes k(x))\Big)^\vee\\&\cong\ker\left(H^0(\mathcal{F}^0\otimes k(x))\to H^0(\mathcal{F}^1\otimes k(x))\right)^\vee.\end{aligned}$$

and a commutative diagram

$$\begin{array}{ccc}\mathbf{H}^2(\mathcal{F}^\bullet\otimes k(x)) & \xrightarrow{\mathrm{Tr}} & \mathbf{H}^2(\Omega^\bullet_{\mathcal{C}_x})\\ \downarrow\cong & & \downarrow\cong\\ \ker\Big(H^0(\mathcal{F}^0\otimes k(x))\xrightarrow{\nabla_{\mathcal{F}^\bullet}}H^0(\mathcal{F}^1\otimes k(x))\Big)^\vee & \xrightarrow{\iota^\vee} & \ker\Big(H^0(\mathcal{O}_{\mathcal{C}_x})\xrightarrow{d}H^0(\Omega^1_{\mathcal{C}_x})\Big)^\vee.\end{array}$$

We can see by the stability that the endomorphisms of $(\tilde{E},\tilde{\nabla},\{\tilde{l}_j^{(i)}\})\otimes k(x)$ are scalar multiplications. So we have $\ker\left(H^0(\mathcal{F}^0\otimes k(x))\to H^0(\mathcal{F}^1\otimes k(x))\right)=\mathbf{C}$ and the canonical inclusion

$$\iota:\ker\Big(H^0(\mathcal{O}_{\mathcal{C}_x})\xrightarrow{d}H^0(\Omega^1_{\mathcal{C}_x})\Big)\longrightarrow\ker\Big(H^0(\mathcal{F}^0\otimes k(x))\xrightarrow{\nabla_{\mathcal{F}^\bullet}}H^0(\mathcal{F}^1\otimes k(x))\Big)$$

is an isomorphism. Hence Tr is an isomorphism and $\omega(E,\nabla,\{l_j^{(i)}\})=0$ because $\mathrm{Tr}(\omega(E,\nabla,\{l_j^{(i)}\}))=0$. So we have proved that $M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)$ is smooth over $T\times\Lambda_r^{(n)}(d)$.

We have a canonical isomorphism $\Theta_{M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)/T\times\Lambda_r^{(n)}(d)}\cong\mathbf{R}^1(\pi_{T\times\Lambda_r^{(n)}(d)})_*(\mathcal{F}^\bullet)$, where $\pi_{T\times\Lambda_r^{(n)}(d)}:\mathcal{C}\to T\times\Lambda_r^{(n)}(d)$ is the projection. From the spectral sequence $H^p(\mathcal{F}^q\otimes k(x))\Rightarrow\mathbf{H}^{p+q}(\mathcal{F}^\bullet\otimes k(x))$, we obtain an exact sequence

$$0\to\mathbf{C}\to H^0(\mathcal{F}^0(x))\to H^0(\mathcal{F}^1(x))\to\mathbf{H}^1(\mathcal{F}^\bullet(x))\to H^1(\mathcal{F}^0(x))\to H^1(\mathcal{F}^1(x))\to\mathbf{C}\to 0$$

for any $x\in M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)$. So we have $\dim\mathbf{H}^1(\mathcal{F}^\bullet(x))=-2\chi(\mathcal{F}^0(x))+2$, because $H^0(\mathcal{F}^1(x))\cong H^1(\mathcal{F}^0(x))^\vee$ and $H^1(\mathcal{F}^1(x))\cong H^0(\mathcal{F}^0(x))^\vee$. Since there is an exact sequence

$$0\to\mathcal{F}^0(x)\to\mathcal{E}nd(\tilde{E}(x))\to\bigoplus_{1\leq i\leq n,0\leq j\leq r-1}\mathrm{Hom}(l_j^{(i)}/l_{j+1}^{(i)},l_{j+1}^{(i)})\to 0.$$

we have

$$\chi(\mathcal{F}^0(x))=\chi(\mathcal{E}nd(\tilde{E}(x)))-nr(r-1)/2=r^2(1-g)-nr(r-1)/2.$$

Thus we can see that every fiber of $M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)$ over $T\times\Lambda_r^{(n)}(d)$ is equidimensional and its dimension is $2r^2(g-1)+nr(r-1)+2$ if it is non-empty. $\square$

In the rest of this section, we will give some remark on elementary transformation and the induced isomorphism on the moduli space of parabolic connections.

Take a point $\boldsymbol{\lambda} \in \Lambda_r^{(n)}(d)$. We can write

$$\prod_{j=0}^{r-1}(x-\lambda_j^{(i)}) = \prod_{k=1}^{s}(x-\mu_k^{(i)})^{m_k^{(i)}},$$

where $\mu_k^{(i)}$ $(1 \leq k \leq s)$ are all mutually distinct, $\mathrm{Re}(\mu_1^{(i)}) > \mathrm{Re}(\mu_2^{(i)}) > \cdots > \mathrm{Re}(\mu_s^{(i)})$ and x is an indeterminate. For any member $(E, \nabla_E, \{l_j^{(i)}\}) \in M_C^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda})$, we can find $j_1^{(k)}, \ldots, j_{m_k^{(i)}}^{(k)}$ for $1 \leq k \leq s$ satisfying $0 \leq j_0^{(k)} < j_1^{(k)} < \cdots < j_{m_k^{(i)}}^{(k)} \leq r$ such that

$$\dim\left(\left(\ker\left(\mathsf{res}_{t_i}(\nabla_E) - \mu_k^{(i)} \cdot \mathrm{id}\right) \cap l_{j_p^{(k)}}^{(i)}\right) \Big/ \left(\ker\left(\mathsf{res}_{t_i}(\nabla_E) - \mu_k^{(i)} \cdot \mathrm{id}\right) \cap l_{j_{p+1}^{(k)}}^{(i)}\right)\right) = 1$$

for $p = 0, \ldots, m_k^{(i)} - 1$. Let $E|_{t_i} = (l')_0^{(i)} \supset (l')_1^{(i)} \supset \cdots \supset (l')_{r-1}^{(i)} \supset (l')_r^{(i)} = 0$ be the filtration satisfying $\dim(l')_p^{(i)}/(l')_{p+1}^{(i)} = 1$ for $p = 0, \ldots, r-1$ and

$$(l')_q^{(i)} \supset \ker(\mathsf{res}_{t_i}(\nabla_E) - \mu_k^{(i)} \cdot \mathrm{id}) \cap l_{j_p^{(k)}}^{(i)}$$

for $q = 0, \ldots, r-1$, where $q = r - (m_k^{(i)} - p) - \sum_{k<k'\leq s} m_{k'}^{(i)}$, $1 \leq k \leq s$ and $0 \leq p \leq m_k^{(i)}$. We put $\nu_j^{(i')} = \lambda_j^{(i')}$ for $i' \neq i$,

$$(\nu_0^{(i)}, \ldots, \nu_{r-1}^{(i)}) := (\overbrace{\mu_1^{(i)}, \ldots, \mu_1^{(i)}}^{m_1^{(i)}}, \ldots, \overbrace{\mu_k^{(i)}, \ldots, \mu_k^{(i)}}^{m_k^{(i)}}, \ldots, \overbrace{\mu_s^{(i)}, \ldots, \mu_s^{(i)}}^{m_s^{(i)}}).$$

and $\boldsymbol{\nu} := (\nu_j^{(i')})_{i',j}$. Then the correspondence $(E, \nabla_E, \{l_j^{(i)}\}) \mapsto (E, \nabla_E, \{(l')_j^{(i)}\})$ determines an isomorphism

$$a_i : \mathcal{M}_C(\mathbf{t}, \boldsymbol{\lambda}) \xrightarrow{\sim} \mathcal{M}_C(\mathbf{t}, \boldsymbol{\nu}), \tag{3}$$

where $\mathcal{M}_C(\mathbf{t}, \boldsymbol{\lambda})$ (resp. $\mathcal{M}_C(\mathbf{t}, \boldsymbol{\nu})$) is the moduli stack of $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connections (resp. $(\mathbf{t}, \boldsymbol{\nu})$-parabolic connections) without stability condition. Next we construct another transformation. For a $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connection $(E, \nabla_E, \{l_j^{(i)}\})$, we put $E' := \ker(E \to E|_{t_i}/l_j^{(i)})$. Then ∇_E induces a connection

$$\nabla_{\tilde{E}} : E' \longrightarrow E' \otimes \Omega_C^1(t_1 + \cdots + t_n).$$

Let $(l')_{r-j}^{(i)} \supset \cdots \supset (l')_r^{(i)} = 0$ be the image of the filtration $E(-t_i)|_{t_i} = l_0^{(i)} \otimes 1 \supset \cdots \supset l_j^{(i)} \otimes 1$ by the linear map $E(-t_i)|_{t_i} \to E'|_{t_i}$ and $E'|_{t_i} = (l')_0^{(i)} \supset \cdots \supset (l')_{r-j}^{(i)}$ be the inverse image of the filtration $l_j^{(i)} \supset \cdots \supset l_r^{(i)}$ by the linear map $E'|_{t_i} \to E|_{t_i}$. We put $\nu_j^{(i')} := \lambda_j^{(i')}$ for $i' \neq i$,

$$(\nu_0^{(i)}, \ldots, \nu_{r-1}^{(i)}) := (\lambda_j^{(i)}, \ldots, \lambda_r^{(i)}, \lambda_0^{(i)} + 1, \ldots, \lambda_{j-1}^{(i)} + 1),$$

and $\boldsymbol{\nu} = (\nu_j^{(i')})$. Then we obtain an isomorphism

$$Elm_{l_j^{(i)}} : \mathcal{M}_C(\mathbf{t}, \boldsymbol{\lambda}) \xrightarrow{\sim} \mathcal{M}_C(\mathbf{t}, \boldsymbol{\nu}) \tag{4}$$

of moduli stacks. The correspondence $(E, \nabla, \{l_j^{(i)}\}) \mapsto (E, \nabla, \{l_j^{(i)}\}) \otimes \mathcal{O}_C(t_i)$ also induces an isomorphism

$$b_i : \mathcal{M}_C(\mathbf{t}, \boldsymbol{\lambda}) \xrightarrow{\sim} \mathcal{M}_C(\mathbf{t}, \boldsymbol{\nu}) \tag{5}$$

where $\nu_j^{(i')} = \lambda_j^{(i')}$ for $i' \neq i$, $\nu_j^{(i)} = \lambda_j^{(i)} - 1$ and $\boldsymbol{\nu} = (\nu_j^{(i')})_{i',j}$.

Looking at the change of $\boldsymbol{\lambda}$ by the transformations $Elm_{l_j^{(i)}}$ and b_i, we can easily see the following proposition, which is useful in considering Riemann-Hilbert correspondence in section 5.

Proposition 3.1. *Composing the isomorphisms a_i, b_i, b_i^{-1} and $Elm_{l_j^{(i)}}$ for $i = 1, \ldots, n$ given in (3), (4) and (5), we obtain an isomorphism*

$$\sigma : \mathcal{M}_C(\mathbf{t}, \boldsymbol{\lambda}) \xrightarrow{\sim} \mathcal{M}_C(\mathbf{t}, \boldsymbol{\mu}) \tag{6}$$

of moduli stacks of parabolic connections, where $0 \leq \mathrm{Re}(\mu^{(i)}_j) < 1$ *for any* i, j.

As a corollary, we obtain the following:

Corollary 3.1. *Take* $\boldsymbol{\lambda} \in \Lambda^{(n)}_r(d)$ *and* $\boldsymbol{\mu} \in \Lambda^{(n)}_r(d')$ *which satisfy* $rh(\boldsymbol{\lambda}) = rh(\boldsymbol{\mu}) \in \mathcal{A}^{(n)}_r$*, where* rh *is the morphism defined in (1). Then we can obtain by composing* $a_i, a_i^{-1}, b_i, b_i^{-1}$ *and* $Elm_{l^{(i)}_j}$ *an isomorphism*

$$\sigma : \mathcal{M}_C(\mathbf{t}, \boldsymbol{\lambda}) \xrightarrow{\sim} \mathcal{M}_C(\mathbf{t}, \boldsymbol{\mu})$$

of moduli stacks of parabolic connections.

Proof. Applying Proposition 3.1, we obtain an isomorphism

$$\sigma : \mathcal{M}_C(\mathbf{t}, \boldsymbol{\lambda}) \xrightarrow{\sim} \mathcal{M}_C(\mathbf{t}, \boldsymbol{\nu})$$

of moduli stacks of parabolic connections, where $0 \leq \mathrm{Re}(\nu^{(i)}_j) < 1$ for any i, j. We may farther assume that $\nu^{(i)}_j \leq \nu^{(i)}_{j'}$ for any i and $j \leq j'$. Applying Proposition 3.1 again, we have an isomorphism

$$\sigma' : \mathcal{M}_C(\mathbf{t}, \boldsymbol{\mu}) \xrightarrow{\sim} \mathcal{M}_C(\mathbf{t}, \boldsymbol{\nu}).$$

Then

$$(\sigma')^{-1} \circ \sigma : \mathcal{M}_C(\mathbf{t}, \boldsymbol{\lambda}) \xrightarrow{\sim} \mathcal{M}_C(\mathbf{t}, \boldsymbol{\mu})$$

is a desired isomorphism. □

4. Irreducibility of the moduli space

In this section we will prove the irreducibility of $M^{\boldsymbol{\alpha}}_{\mathcal{C}_x}(\tilde{\mathbf{t}}_x, \boldsymbol{\lambda})$, which is the fiber of $M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d)$ over $(x, \boldsymbol{\lambda}) \in T \times \Lambda^{(n)}_r(d)$. We simply denote $\mathcal{C}_x$ by C, $\tilde{\mathbf{t}}_x$ by $\mathbf{t} = (t_1, \ldots, t_n)$ and $t_1 + \cdots + t_n$ by $D(\mathbf{t})$. First consider the morphism

$$\begin{aligned} \det : M^{\boldsymbol{\alpha}}_C(\mathbf{t}, \boldsymbol{\lambda}) &\longrightarrow M_C(\mathbf{t}, \det(\boldsymbol{\lambda})) \\ (E, \nabla, \{l^{(i)}_j\}) &\mapsto (\det E, \det(\nabla)), \end{aligned} \tag{7}$$

where $\det(\nabla)$ is the logarithmic connection on $\det E$ induced by ∇ and $\det(\boldsymbol{\lambda}) = (\sum_{j=0}^{r-1} \lambda^{(i)}_j)^{1 \leq i \leq n} \in \Lambda^{(n)}_1(d)$. Note that a line bundle with a connection is always stable. We can also construct $M_C(\mathbf{t}, \det(\boldsymbol{\lambda}))$ as an affine space bundle over Pic^d_C whose fiber is of dimension

$$h^0(\mathcal{E}nd(\det E) \otimes \Omega^1_C) = h^0(\Omega^1_C) = g.$$

Thus $M_C(\mathbf{t}, \det(\boldsymbol{\lambda}))$ is a smooth irreducible variety of dimension $2g$.

We can prove the smoothness of the morphism (7). Indeed let A be an artinian local ring over $M_C(\mathbf{t}, \det(\boldsymbol{\lambda}))$ with maximal ideal m, I an ideal of A satisfying $mI = 0$ and take any A/I-valued point $(E, \nabla, \{l^{(i)}_j\})$ of $M^{\boldsymbol{\alpha}}_C(\mathbf{t}, \boldsymbol{\lambda})$. We write $(\bar{E}, \bar{\nabla}, \{\bar{l}^{(i)}_j\}) := (E, \nabla, \{l^{(i)}_j\}) \otimes A/m$ and put

$$\mathcal{F}^0_0 := \left\{ s \in \mathcal{E}nd(\bar{E}) \,\middle|\, \mathrm{Tr}(s) = 0 \text{ and } s(t_i)(\bar{l}^{(i)}_j) \subset \bar{l}^{(i)}_j \text{ for any } i, j \right\}$$
$$\mathcal{F}^1_0 := \left\{ s \in \mathcal{E}nd(\bar{E}) \otimes \Omega^1_X(D(\mathbf{t})) \,\middle|\, \mathrm{Tr}(s) = 0 \text{ and } s(t_i)(\bar{l}^{(i)}_j) \subset \bar{l}^{(i)}_{j+1} \text{ for any } i, j \right\}$$

We define a complex $\mathcal{F}^\bullet_0$ by

$$\nabla_{\mathcal{F}^\bullet_0} : \mathcal{F}^0_0 \ni s \mapsto \bar{\nabla} \circ s - s \circ \bar{\nabla} \in \mathcal{F}^1_0.$$

Then the obstruction class for the lifting of $(E, \nabla, \{l^{(i)}_j\})$ to an A-valued point of $M^{\boldsymbol{\alpha}}_C(\mathbf{t}, \boldsymbol{\lambda})$ lies in $\mathbf{H}^2(\mathcal{F}^\bullet_0)$. We can see by the same way as the proof of Theorem 2.1 that

$$\mathbf{H}^2(\mathcal{F}^\bullet_0) \cong \left(\ker(H^0(\mathcal{F}^0_0) \to H^0(\mathcal{F}^1_0))\right)^\vee$$

and we can see by the stability of $(\bar{E}, \bar{\nabla}, \{\bar{l}^{(i)}_j\})$ that $\ker(H^0(\mathcal{F}^0_0) \to H^0(\mathcal{F}^1_0)) = 0$. Hence the morphism det in (7) is smooth.

From the above argument, it is sufficient to prove the irreducibility of the fibers of the morphism (7) $\det : M^{\boldsymbol{\alpha}}_C(\mathbf{t}, \boldsymbol{\lambda}) \to M_C(\mathbf{t}, \det(\boldsymbol{\lambda}))$ in order to obtain the irreducibility of $M^{\boldsymbol{\alpha}}_C(\mathbf{t}, \boldsymbol{\lambda})$. So we fix a line bundle L on C and a connection

$$\nabla_L : L \longrightarrow L \otimes \Omega^1_C(t_1 + \cdots + t_n).$$

We set

$$M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda},L):=\left\{(E,\nabla,\{l^{(i)}_j\})\in M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda})\,\middle|\,(\det E,\det(\nabla))\cong(L,\nabla_L)\right\}.$$

Then we can see that $M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda},L)$ is smooth of equidimension $(r-1)(2(r+1)(g-1)+rn)$.

Proposition 4.1. *Assume that $g\geq 2$ and $n\geq 1$. Then $M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda},L)$ is an irreducible variety of dimension $(r-1)(2(r+1)(g-1)+rn)$.*

Proof. By taking elementary transform, we can obtain an isomorphism

$$\sigma:\mathcal{M}_C(\mathbf{t},\boldsymbol{\lambda},L)\xrightarrow{\sim}\mathcal{M}_C(\mathbf{t},\boldsymbol{\lambda}',L')$$

of moduli stacks of parabolic connections without stability condition, where r and $\deg L'$ are coprime. Consider the open subscheme

$$N:=\left\{(E,\nabla,\{l^{(i)}_j\})\in M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda},L)\,\middle|\,\sigma(E)\text{ is a stable vector bundle}\right\}$$

of $M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda},L)$. We will see that N is irreducible. Let $M_C(r,L')$ be the moduli space of stable vector bundles of rank r with the determinant L'. As is well-known, $M_C(r,L')$ is a smooth irreducible variety and there is a universal bundle $\mathcal{E}$ on $C\times M_C(r,L')$. Note that there is a line bundle $\mathcal{L}$ on $M_C(r,L')$ such that $\det(\mathcal{E})\cong L\otimes\mathcal{L}$. We can parameterize the parabolic structures on $\mathcal{E}_s$ $(s\in M_C(r,L'))$ by a product of flag schemes

$$U:=\prod_{i=1}^n\mathrm{Flag}(\mathcal{E}|_{t_i\times M_C(r,L')}),$$

which is obviously smooth and irreducible. Let $\{\tilde{l}^{(i)}_j\}$ be the universal family over U, where

$$\mathcal{E}_U|_{t_i\times U}=\tilde{l}^{(i)}_0\supset\tilde{l}^{(i)}_1\supset\cdots\supset\tilde{l}^{(i)}_{r-1}\supset\tilde{l}^{(i)}_r=0$$

is the filtration by subbundles for $i=1,\ldots,n$ such that $\dim(\tilde{l}^{(i)}_j)_s/(\tilde{l}^{(i)}_{j+1})_s=1$ for any i,j and $s\in U$. Consider the functor $\mathrm{Conn}_U:(Sch/U)\to(Sets)$ defined by

$$\mathrm{Conn}_U(S)=\left\{\nabla:\mathcal{E}_S\to\mathcal{E}_S\otimes\Omega^1_C(D(\mathbf{t}))\,\middle|\,\begin{array}{l}\nabla\text{ is a relative connection satisfying}\\ (\mathsf{res}_{t_i\times S}(\nabla)-\lambda^{(i)}_j)(\tilde{l}^{(i)}_j)_S\subset(\tilde{l}^{(i)}_{j+1})_S\text{ for any }i,j\\ \text{and }\det(\nabla)=(\nabla_L)_S\otimes\mathcal{L}_S\end{array}\right\}.$$

We can define a morphism of functors

$$\begin{aligned}&\mathrm{Conn}_U\longrightarrow\mathrm{Quot}_{\Lambda^1_{D(\mathbf{t})}\otimes\mathcal{E}/C\times U/U}\\&\nabla\mapsto\left[\Lambda^1_{D(\mathbf{t})}\otimes\mathcal{E}_U\ni(a,v)\otimes e\mapsto ae+\nabla_v(e)\in\mathcal{E}_U\right],\end{aligned}$$

which is representable by an immersion. So there is a locally closed subscheme Y of $\mathrm{Quot}_{\Lambda^1_{D(\mathbf{t})}\otimes\mathcal{E}/C\times U/U}$ which represents the functor Conn_U. We will show that Y is smooth over U. So let $\tilde{\nabla}:\mathcal{E}_Y\to\mathcal{E}_Y\otimes\Omega^1_C(D(\mathbf{t}))$ be the universal connection and put

$$\begin{aligned}\mathcal{F}^0_0&:=\left\{s\in\mathcal{E}nd(\mathcal{E})\,\middle|\,\mathrm{Tr}(s)=0\text{ and }s|_{t_i\times Y}((\tilde{l}^{(i)}_j)_Y)\subset(\tilde{l}^{(i)}_j)_Y\text{ for any }i,j\right\}\\ \mathcal{F}^1_0&:=\left\{s\in\mathcal{E}nd(\mathcal{E})\otimes\Omega^1_C(D(\mathbf{t}))\,\middle|\,\mathrm{Tr}(s)=0\text{ and }\mathsf{res}_{t_i\times Y}(s)((\tilde{l}^{(i)}_j)_Y)\subset(\tilde{l}^{(i)}_{j+1})_Y\text{ for any }i,j\right\}\\ \nabla_{\mathcal{F}^\bullet_0}&:\mathcal{F}^0_0\ni s\mapsto\tilde{\nabla}\circ s-s\circ\tilde{\nabla}\in\mathcal{F}^1_0.\end{aligned}$$

Let A be an aritinian local ring over U with maximal ideal m and I an ideal of A satisfying $mI=0$. Take any A/I-valued point x of Y over U. We put $\bar{x}:=x\otimes A/m$. Then the obstruction class for the lifting of x to an A-valued point of Y lies in $H^1(\mathcal{F}^1_0\otimes k(\bar{x}))\otimes I$. We can see that $H^1(\mathcal{F}^1_0\otimes k(\bar{x}))\cong H^0(\mathcal{F}^0_0\otimes k(\bar{x}))^\vee=0$ because $\mathcal{E}\otimes k(\bar{x})$ is a stable bundle. Thus Y is smooth over U. We can see that the fiber Y_s of any point $s\in U$ is isomorphic to the affine space isomorphic to $H^0(\mathcal{F}^1_0\otimes k(s))$. So Y is irreducible. Consider the open subscheme

$$Y':=\left\{x\in Y\,\middle|\,\sigma^{-1}(\mathcal{E}_x,\tilde{\nabla}\otimes k(x),\{(\tilde{l}^{(i)}_j)_x\})\text{ is }\boldsymbol{\alpha}\text{-stable}\right\}$$

of Y. Then we obtain a morphism $Y'\to M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda},L)$ whose image is just N. Thus N is irreducible.

For the proof of irreducibility of $M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda},L)$, it is sufficient to show that $\dim(M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda},L)\setminus N)<\dim M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda},L)=(r-1)(2(r+1)(g-1)+rn)$. Take any member $(E,\nabla_E,\{l^{(i)}_j\})\in\sigma(M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda},L)\setminus N)$. Let

$$0=E_0\subset E_1\subset E_2\subset\cdots\subset E_s=E$$

be the Harder-Narasimhan filtraion of E. Note that $s>1$ because r and $\deg E=\deg L'$ are coprime. Put $\bar{E}_k:=E_k/E_{k-1}$ for $k=1,\ldots,s$. For each $\bar{E}_k$, there is a Jordan-Hölder filtration

$$0=E^{(0)}_k\subset E^{(1)}_k\subset E^{(2)}_k\subset\cdots\subset E^{(m_k)}_k=\bar{E}_k.$$

We put $\bar{E}^{(i)}_k:=E^{(i)}_k/E^{(i-1)}_k$ for $i=1,\ldots,m_k$. Put $r^{(j)}_k=\operatorname{rank}\bar{E}^{(j)}_k$ and $d^{(j)}_k=\deg\bar{E}^{(j)}_k$. We can parameterize each $\bar{E}^{(j)}_k$ by the moduli space $M(r^{(j)}_k,d^{(j)}_k)$ of stable bundles of rank $r^{(j)}_k$ and degree $d^{(j)}_k$ whose dimension is $\dim\operatorname{Ext}^1(\bar{E}^{(j)}_k,\bar{E}^{(j)}_k)$. Replacing $M(r^{(j)}_k,d^{(j)}_k)$ by an étale covering, we may assume that there is a universal bundle $\bar{\mathcal{E}}^{(j)}_k$ on $C\times M(r^{(j)}_k,d^{(j)}_k)$. If we put

$$X:=\left\{(x^{(j)}_k)\in\prod_{1\leq k\leq s,1\leq j\leq m_k}M(r^{(j)}_k,d^{(j)}_k)\;\middle|\;\bigotimes_{j,k}\det\left(\bar{\mathcal{E}}^{(j)}_k\right)_{x^{(j)}_k}\cong L\right\},$$

then X is smooth of dimension $-g+\sum_{k=1}^s\sum_{j=1}^{m_k}\dim\operatorname{Ext}^1(\bar{E}^{(j)}_k,\bar{E}^{(j)}_k)$. Replacing X by its stratification, we may assume that the relative Ext-sheaf $\operatorname{Ext}^1_{C\times X/X}(\bar{\mathcal{E}}^{(2)}_k,\bar{\mathcal{E}}^{(1)}_k)$ becomes locally free. We put $\tilde{W}^{(2)}_k:=X$ and $\mathcal{E}^{(2)}_k:=\bar{\mathcal{E}}^{(1)}_k\oplus\bar{\mathcal{E}}^{(2)}_k$ if the extension

$$0\longrightarrow\bar{E}^{(1)}_k\longrightarrow E^{(2)}_k\longrightarrow\bar{E}^{(2)}_k\longrightarrow 0$$

splits. Otherwise we put

$$\tilde{W}^{(2)}_k:=\mathbf{P}_*\left(\operatorname{Ext}^1_{C\times X/X}(\bar{\mathcal{E}}^{(2)}_k,\bar{\mathcal{E}}^{(1)}_k)\right)$$

and take a universal extension

$$0\longrightarrow(\bar{\mathcal{E}}^{(1)}_k)_{\tilde{W}^{(2)}_k}\otimes\mathcal{L}^{(2)}_k\longrightarrow\mathcal{E}^{(2)}_k\longrightarrow(\bar{\mathcal{E}}^{(2)}_k)_{\tilde{W}^{(2)}_k}\longrightarrow 0$$

for some line bundle $\mathcal{L}^{(2)}_k$ on $\tilde{W}^{(2)}_k$. We define $\tilde{W}^{(j)}_k$ and $\mathcal{E}^{(j)}_k$ inductively on j as follows: Replacing $\tilde{W}^{(j-1)}_k$ by its stratification, we may assume that the relative Ext-sheaf $\operatorname{Ext}^1_{C\times\tilde{W}^{(j-1)}_k/\tilde{W}^{(j-1)}_k}((\bar{\mathcal{E}}^{(j)}_k)_{\tilde{W}^{(j-1)}_k},\mathcal{E}^{(j-1)}_k)$ is locally free. Then put $\tilde{W}^{(j)}_k:=\tilde{W}^{(j-1)}_k$ and $\mathcal{E}^{(j)}_k:=\mathcal{E}^{(j-1)}_k\oplus(\bar{\mathcal{E}}^{(j)}_k)_{\tilde{W}^{(j)}_k}$ if the extension

$$0\longrightarrow E^{(j-1)}_k\longrightarrow E^{(j)}_k\longrightarrow\bar{E}^{(j)}_k\longrightarrow 0$$

splits and otherwise we put

$$\tilde{W}^{(j)}_k:=\mathbf{P}_*\left(\operatorname{Ext}^1_{C\times\tilde{W}^{(j-1)}_k/\tilde{W}^{(j-1)}_k}((\bar{\mathcal{E}}^{(j)}_k)_{\tilde{W}^{(j-1)}_k},\mathcal{E}^{(j-1)}_k)\right)$$

and take a universal extension

$$0\longrightarrow(\mathcal{E}^{(j-1)}_k)_{\tilde{W}^{(j)}_k}\otimes\mathcal{L}^{(j)}_k\longrightarrow\mathcal{E}^{(j)}_k\longrightarrow(\bar{\mathcal{E}}^{(j)}_k)_{\tilde{W}^{(j)}_k}\longrightarrow 0$$

for some line bundle $\mathcal{L}^{(j)}_k$ on $\tilde{W}^{(j)}_k$. We can see by the construction that the relative dimension of $\tilde{W}^{(m_k)}_k$ over X at the point corresponding to the extensions

$$0\longrightarrow E^{(j-1)}_k\longrightarrow E^{(j)}_k\longrightarrow\bar{E}^{(j)}_k\longrightarrow 0\quad(j=2,\ldots,m_k,\ E^{(1)}_k=\bar{E}^{(1)}_k)$$

is at most $1-m_k+\sum_{1\leq i<j\leq m_k}\dim\operatorname{Ext}^1(\bar{E}^{(j)}_k,\bar{E}^{(i)}_k)$. We put

$$W:=\tilde{W}^{(m_1)}_1\times_X\cdots\times_X\tilde{W}^{(m_s)}_s$$

and $\bar{\mathcal{E}}_k:=(\mathcal{E}^{(m_k)}_k)_W$ for $k=1,\ldots,s$. Replacing W by its stratification, we assume that the relative Ext-sheaf $\operatorname{Ext}^1_{C\times W/W}(\bar{\mathcal{E}}_2,\bar{\mathcal{E}}_1)$ is locally free. Then we put $W_2:=W$ and $\mathcal{E}_2:=\bar{\mathcal{E}}_1\oplus\bar{\mathcal{E}}_2$ if the extension

$$0\longrightarrow\bar{E}_1\longrightarrow E_2\longrightarrow\bar{E}_2\longrightarrow 0$$

splits and otherwise we put

$$W_2:=\mathbf{P}_*\left(\operatorname{Ext}^1_{C\times W/W}(\bar{\mathcal{E}}_2,\bar{\mathcal{E}}_1)\right)$$

and take a universal extension

$$0 \longrightarrow (\bar{\mathcal{E}}_1)_{W_2} \otimes \mathcal{L}_2 \longrightarrow \mathcal{E}_2 \longrightarrow (\bar{\mathcal{E}}_2)_{W_2} \longrightarrow 0$$

for some line bundle $\mathcal{L}_2$. We define W_k and $\mathcal{E}_k$ inductively as follows: Replacing W_{k-1} by its stratification, we assume that the relative Ext-sheaf $\mathrm{Ext}^1_{C\times W_{k-1}/W_{k-1}}((\bar{\mathcal{E}}_k)_{W_{k-1}}, \mathcal{E}_{k-1})$ is locally free. Then we put $W_k := W_{k-1}$ and $\mathcal{E}_k := \bar{\mathcal{E}}_k \oplus \mathcal{E}_{k-1}$ if the extension

$$0 \longrightarrow E_{k-1} \longrightarrow E_k \longrightarrow \bar{E}_k \longrightarrow 0$$

splits and otherwise we put

$$W_k := \mathbf{P}_* \left(\mathrm{Ext}^1_{C\times W_{k-1}/W_{k-1}}((\bar{\mathcal{E}}_k)_{W_{k-1}}, \mathcal{E}_{k-1}) \right)$$

and take a universal extension

$$0 \longrightarrow (\mathcal{E}_{k-1})_{W_k} \otimes \mathcal{L}_k \longrightarrow \mathcal{E}_k \longrightarrow (\bar{\mathcal{E}}_k)_{W_k} \longrightarrow 0$$

for some line bundle $\mathcal{L}_k$ on W_k. Then we can see that the dimension of W_s at the point corresponding to the extensions

$$0 = E_0 \subset E_1 \subset \cdots E_s = E, \quad \bar{E}_k = E_k/E_{k-1}$$
$$0 = E_k^{(0)} \subset E_k^{(1)} \subset \cdots \subset E_k^{(m_k)} = \bar{E}_k, \quad \bar{E}_k^{(j)} = E_k^{(j)}/E_k^{(j-1)}$$

is at most

$$\begin{aligned}
&-g + \sum_{k=1}^{s}\sum_{j=1}^{m_k} \dim \mathrm{Ext}^1(\bar{E}_k^{(j)}, \bar{E}_k^{(j)}) + \sum_{k=1}^{s} \sum_{1\le i<j\le m_k} \left(1 - m_k + \dim \mathrm{Ext}^1(\bar{E}_k^{(j)}, \bar{E}_k^{(i)})\right) \\
&\quad + 1 - s + \sum_{k<k'} \sum_{1\le i\le m_{k'}, 1\le j\le m_k} \dim \mathrm{Ext}^1(\bar{E}_{k'}^{(i)}, \bar{E}_k^{(j)}) \\
\le& \sum_{k=1}^{s}\sum_{i=1}^{m_k}((r_k^{(i)})^2(g-1)+1) + \sum_{k=1}^{s}\left(1 - m_k + \sum_{1\le i<j\le m_k}(r_k^{(i)} r_k^{(j)}(g-1)+1)\right) \\
&\quad - g + 1 - s + \sum_{k<k'} \sum_{1\le i\le m_k, 1\le j\le m_{k'}} (r_k^{(i)} r_{k'}^{(j)}(g-1) + r_k^{(i)} r_{k'}^{(j)} - 1) \\
<& \sum_{k=1}^{s}\sum_{i=1}^{m_k}(r_k^{(i)})^2(g-1) + \sum_{k=1}^{s}\sum_{1\le i<j\le m_k} 2 r_k^{(i)} r_k^{(j)}(g-1) \\
&\quad - g + 1 + \sum_{k<k'} \sum_{1\le i\le m_k, 1\le j\le m_{k'}} 2 r_k^{(i)} r_{k'}^{(j)}(g-1) \\
=&(r^2-1)(g-1)
\end{aligned}$$

because we have $s > 1$. Consider the product of flag schemes

$$Z := \prod_{i=1}^{n} \mathrm{Flag}(\mathcal{E}|_{t_i\times W_s})$$

over W_s and put $\mathcal{E} := (\mathcal{E}_s)_Z$. Then there is a universal parabolic structure

$$\mathcal{E}|_{t_i\times Z} = \tilde{l}_0^{(i)} \supset \tilde{l}_1^{(i)} \supset \cdots \supset \tilde{l}_{r-1}^{(i)} \supset \tilde{l}_r^{(i)} = 0.$$

The dimension of Z at the point corresponding to the extensions

$$0 = E_0 \subset E_1 \subset \cdots E_s = E, \quad \bar{E}_k = E_k/E_{k-1}$$
$$0 = E_k^{(0)} \subset E_k^{(1)} \subset \cdots \subset E_k^{(m_k)} = \bar{E}_k, \quad \bar{E}_k^{(j)} = E_k^{(j)}/E_k^{(j-1)}$$

and the parabolic structure

$$E|_{t_i} = l_0^{(i)} \supset l_1^{(i)} \supset \cdots \supset l_{r-1}^{(i)} \supset l_r^{(i)} = 0 \quad (i = 1, \dots, n)$$

is less than $(r^2-1)(g-1) + r(r-1)n/2 = (r-1)((r+1)(g-1) + rn/2)$.

Consider the functor $\mathrm{Conn}_Z : (Sch/Z) \to (Sets)$ defined by

$$\mathrm{Conn}_Z(S) = \left\{ \nabla : \mathcal{E}_S \to \mathcal{E}_S \otimes \Omega^1_C(D(\mathbf{t})) \;\middle|\; \begin{array}{l} \nabla \text{ is a connection such that} \\ (\mathrm{res}_{t_i \times S}(\nabla) - \lambda^{(i)}_j)((\tilde{l}^{(i)}_j)_S) \subset (\tilde{l}^{(i)}_{j+1})_S \text{ for any } i, j \\ \text{and } \det(\nabla) = \nabla_L \otimes \mathcal{L} \text{ via an identification} \\ \mathcal{E} = L \otimes \mathcal{L} \text{ for some line bundle } \mathcal{L} \text{ on } S \end{array} \right\}.$$

We can see that Conn_Z can be represented by a scheme B of finite type over Z. Let

$$\tilde{\nabla} : \mathcal{E}_B \longrightarrow \mathcal{E}_B \otimes \Omega^1_C(D(\mathbf{t}))$$

be the universal connection. If we put

$$\mathcal{F}^0_0 := \left\{ u \in \mathcal{E}nd(\mathcal{E}) \;\middle|\; \mathrm{Tr}(u) = 0 \text{ and } u|_{t_i \times Y}((\tilde{l}^{(i)}_j)_Y) \subset (\tilde{l}^{(i)}_j)_Y \text{ for any } i, j \right\}$$

$$\mathcal{F}^1_0 := \left\{ u \in \mathcal{E}nd(\mathcal{E}) \otimes \Omega^1_C(D(\mathbf{t})) \;\middle|\; \mathrm{Tr}(u) = 0 \text{ and } \mathrm{res}_{t_i \times Y}(u)((\tilde{l}^{(i)}_j)_Y) \subset (\tilde{l}^{(i)}_{j+1})_Y \text{ for any } i, j \right\},$$

then the fiber B_x of B over $x \in Z$ is isomorphic to an affine space of dimension $h^0(\mathcal{F}^1_0 \otimes k(x))$. Put

$$B' := \left\{ x \in B \;\middle|\; \sigma^{-1}\left(\mathcal{E}_x, \tilde{\nabla} \otimes k(x), \{\tilde{l}^{(i)}_j \otimes k(x)\right) \text{ is } \boldsymbol{\alpha}\text{-stable} \right\}.$$

Then there is a canonical morphism

$$\psi : B' \longrightarrow M^{\boldsymbol{\alpha}}_C(\mathbf{t}, \boldsymbol{\lambda}, L) \setminus N.$$

Since the dimension of the auotomorphism group of the parabolic bundle $\left(\mathcal{E} \otimes k(x), \{\tilde{l}^{(i)}_j \otimes k(x)\}\right)$ is $1 + h^0(\mathcal{F}^0_0 \otimes k(x))$ for $x \in B'$ and $\dim \mathrm{coker}(H^0(\mathcal{F}^0_0 \otimes k(x)) \to H^0(\mathcal{F}^1_0 \otimes k(x))) = (r-1)((r+1)(g-1)+rn/2)$, we can see that $\dim \psi(B') < (r-1)(2(r+1)(g-1)+rn)$. Note that $M^{\boldsymbol{\alpha}}_C(\mathbf{t}, \boldsymbol{\lambda}, L) \setminus N$ can be covered by a finite union of such $\psi(B')$'s. So we have $\dim(M^{\boldsymbol{\alpha}}_C(\mathbf{t}, \boldsymbol{\lambda}, L) \setminus N) < (r-1)(2(r+1)(g-1)+rn)$. Thus the moduli space $M^{\boldsymbol{\alpha}}_C(\mathbf{t}, \boldsymbol{\lambda}, L)$ is irreducible because it is smooth of equidimension $(r-1)(2(r+1)(g-1)+rn)$. □

Proposition 4.2. *Assume that $g = 1$ and $n \geq 2$. Then $M^{\boldsymbol{\alpha}}_C(\mathbf{t}, \boldsymbol{\lambda}, L)$ is an irreducible variety of dimension $r(r-1)n$.*

Proof. Composing certain elementary transforms, we obtain an isomorphism

$$\sigma : \mathcal{M}_C(\mathbf{t}, \boldsymbol{\lambda}, L) \xrightarrow{\sim} \mathcal{M}_C(\mathbf{t}, \boldsymbol{\lambda}', L'),$$

where r and $\deg L'$ are coprime. Put

$$N := \left\{ (E, \nabla, \{l^{(i)}_j\}) \in M^{\boldsymbol{\alpha}}_C(\mathbf{t}, \boldsymbol{\lambda}, L) \;\middle|\; \sigma(E) \text{ is a stable bundle} \right\}.$$

As in the proof of the previous proposition, we can show that N is irreducible. So it suffices to show that $\dim M^{\boldsymbol{\alpha}}_C(\mathbf{t}, \boldsymbol{\lambda}, L) \setminus N < r(r-1)n$ because $M^{\boldsymbol{\alpha}}_C(\mathbf{t}, \boldsymbol{\lambda}, L)$ is smooth of equidimension $r(r-1)n$. Take any member $(E, \nabla, \{l^{(i)}_j\}) \in \sigma(M^{\boldsymbol{\alpha}}_C(\mathbf{t}, \boldsymbol{\lambda}, L) \setminus N)$. Let $F_s \subset F_{s-1} \subset \cdots \subset F_1 = E$ be the Harder-Narasimhan filtration of E. Note that $s > 1$. We can inductively see that the extension

$$0 \longrightarrow F_p/F_{p+1} \longrightarrow E/F_{p+1} \longrightarrow E/F_p \longrightarrow 0$$

must split for $p = 2, \ldots, s$, where we put $F_{s+1} = 0$. Then we have a decomposition $E \cong E_1 \oplus \cdots \oplus E_s$, where each E_p is semistable and $\mu(E_1) < \mu(E_2) < \cdots < \mu(E_s)$. We can write $E_p = \bigoplus_{j=1}^{r_p} F_{p,j}$, where each $F_{p,j}$ is a succesive extension of a stable bundle $F^{(j)}_p$ and $F^{(j)}_p \not\cong F^{(k)}_p$ for $j \neq k$. We can see that $F_{p,j} \cong G_1 \oplus \cdots \oplus G_m$, where each G_i is a succesive non-split extension of $F^{(j)}_p$. We put $r^{(j)}_p := \mathrm{rank}\, F^{(j)}_p$ and $d^{(j)}_p := \mathrm{rank}\, F^{(j)}_p$. Let $M_C(r^{(j)}_p, d^{(j)}_p)$ be the moduli space of stable bundles on C of rank $r^{(j)}_p$ and degree $d^{(j)}_p$. Replacing $M_C(r^{(j)}_p, d^{(j)}_p)$ by an étale covering, we may assume that there is a universal bundle $\mathcal{F}^{(j)}_p$ on $C \times M_C(r^{(j)}_p, d^{(j)}_p)$. We can construct non-split extensions $\mathcal{G}_1, \ldots, \mathcal{G}_m$ of $\mathcal{F}^{(j)}_p$ such that $F_{p,j} \cong (\mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_m) \otimes k(x_0)$ for the point x_0 of $M_C(r^{(j)}_p, d^{(j)}_p)$ corresponding to $F^{(j)}_p$. We put $\mathcal{F}_{p,j} := \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_m$ and

$$W := \left\{ (x^{(j)}_p) \in \prod_{1 \leq p \leq s, 1 \leq j \leq r_p} M_C(r^{(j)}_p, d^{(j)}_p) \;\middle|\; \bigotimes_{p,j} \det\left(\mathcal{F}_{p,j} \otimes k(x^{(j)}_p)\right) \cong L \right\}.$$

Then $\dim W = -1 + \sum_{p=1}^{s} r_p$. We put $\mathcal{E} := \bigoplus_{j,p} \mathcal{F}_{p,j}$. Then parabolic structure can be parameterized by the product of flag schemes

$$U := \prod_{i=1}^{n} \mathrm{Flag}(\mathcal{E}|_{t_i \times W})$$

over W. The relative dimension of U over W is $r(r-1)n/2$. Let $\{\tilde{l}^{(i)}_j\}$ be the universal parabolic structure on $\mathcal{E}_U$. Note that

$$\mathrm{End}(\mathcal{E} \otimes k(x)) = \bigoplus_{p=1}^{s} \bigoplus_{j=1}^{r_p} \mathrm{End}(\mathcal{F}_{p,j} \otimes k(x)) \oplus \bigoplus_{p<q} \bigoplus_{j,k} \mathrm{Hom}(\mathcal{F}_{p,j} \otimes k(x), \mathcal{F}_{q,k} \otimes k(x))$$

for a point $x \in W$ and the group

$$G_x := \left\{ \sum_{p,j} c_{pj} \mathrm{id}_{\mathcal{F}_{p,j} \otimes k(x)} + \sum_{p<q} \sum_{j,k} a^{(p,q)}_{j,k} \in \mathrm{End}(\mathcal{E} \otimes k(x)) \,\middle|\, \begin{array}{l} c_{pj} \in \mathbf{C}^*, \\ a^{(p,q)}_{j,k} \in \mathrm{Hom}\left(\mathcal{F}_{p,j} \otimes k(x), \mathcal{F}_{q,k} \otimes k(x)\right) \end{array} \right\}$$

acts on the fiber U_x of U over x. Note that we can take a section $a \in \mathrm{Hom}(\mathcal{F}_{p,j} \otimes k(x), \mathcal{F}_{q,k} \otimes k(x))$ with $a(t_1) \neq 0$ or $a(t_2) \neq 0$ if $p < q$. Then modulo the action of G_x, $\tilde{l}^{(1)}_{r-1} \otimes k(y)$ and $\tilde{l}^{(2)}_{r-1} \otimes k(y)$ ($y \in U_x$) can be parameterized by an algebraic scheme whose dimension is at most $2r - \sum_{p=1}^{s} r_p - 2$. So we can take a finite number of subschemes $Y_1, \ldots, Y_l$ of U such that $\dim Y_i \leq -1 + \sum_{p=1}^{s} r_p + r(r-1)n/2 - \sum_{p=1}^{s} r_p = r(r-1)n/2 - 1$ and $\bigcup_{i=1}^{l} G_x(Y_i)_x = U_x$ for any point $x \in W$. We put $Y := \coprod_{i=1}^{l} Y_i$. Replacing Y by its stratification, we may assume that the dimension of

$$F^0_y := \left\{ s \in \mathrm{End}(\mathcal{E}_y) \,\middle|\, \mathrm{Tr}(s) = \text{ and } s(t_i)(\tilde{l}^{(i)}_j)_y \subset (\tilde{l}^{(i)}_j)_y \text{ for each } i, j \right\}$$

is constant for $y \in Y$. There exists a scheme Z over Y such that for any Y-scheme S, we have

$$Z(S) = \left\{ \nabla : \mathcal{E}_S \to \mathcal{E}_S \otimes \Omega^1_C(t_1 + \cdots + t_n) \,\middle|\, \begin{array}{l} \nabla \text{ is a relative connection such that} \\ (\mathsf{res}_{t_i \times S}(\nabla) - \lambda^{(i)}_j)((\tilde{l}^{(i)}_j)_S) \subset (\tilde{l}^{(i)}_{j+1})_S \text{ for any } i, j \\ \text{and } \det(\nabla)) = (\nabla_L) \otimes \mathcal{L} \end{array} \right\}$$

We can see that each fiber of $Z \to Y$ over $y \in Y$ is an affine space isomorphic to

$$F^1_y = \left\{ s \in \mathrm{Hom}(\mathcal{E}_y, \mathcal{E}_y \otimes \Omega^1_C(t_1 + \cdots + t_n)) \,\middle|\, \mathrm{Tr}(s) = 0 \text{ and } \mathsf{res}_{t_i}(s)(\tilde{l}^{(i)}_j)_y \subset (\tilde{l}^{(i)}_{j+1})_y \text{ for any } i, j \right\}.$$

Let $\tilde{\nabla} : \mathcal{E}_Z \to \mathcal{E}_Z \otimes \Omega^1_C(t_1 + \cdots + t_n)$ be the universal relative connection. If we put

$$Z^s := \left\{ y \in Z \,\middle|\, \sigma^{-1}\left((\mathcal{E}, \tilde{\nabla}, \{\tilde{l}^{(i)}_j\}) \otimes k(y) \right) \text{ is } \boldsymbol{\alpha}\text{-stable} \right\},$$

then there is a natural morphism $f : Z^s \to M^{\boldsymbol{\alpha}}_C(\mathbf{t}, \boldsymbol{\lambda}, L)$. For each $z \in Z$ consider the homomorphism

$$\nabla^1_z : F^0_y \otimes k(z) \longrightarrow F^1_y \otimes k(z); \quad \nabla^1_z(s) := \tilde{\nabla}_z \circ s - s \circ \tilde{\nabla}_z.$$

Then the group Stab_y of automorphisms g of the parabolic bundles $(\mathcal{E}_y, \{(\tilde{l}^{(i)}_j)_y\})$ acts on the fiber Z_y and the tangent map of the morphism $\mathrm{Stab}_y \ni g \mapsto gz \in Z^s$ is just the homomorphism ∇^1_z which is injective. Thus the dimension of the image $f(Z^s)$ is at most

$$\dim F^1_y - \dim F^0_y + \dim Y = r(r-1)n/2 + \dim Y \leq r(r-1)n - 1.$$

Since $M^{\boldsymbol{\alpha}}_C(\mathbf{t}, \boldsymbol{\lambda}, L) \setminus N$ is a finite union of such $f(Y)$'s, we have $\dim(M^{\boldsymbol{\alpha}}_C(\mathbf{t}, \boldsymbol{\lambda}, L) \setminus N) < r(r-1)n$. □

Proposition 4.3. *Assume that $rn - 2(r+1) > 0$ and $r \geq 2$. Then $M^{\boldsymbol{\alpha}}_{\mathbf{P}^1}(\mathbf{t}, \boldsymbol{\lambda}, L)$ is an irreducible variety of dimension $(r-1)(rn - 2r - 2)$.*

Proof. First we will show that the Zariski open set

$$N := \left\{ (E, \nabla, \{l^{(i)}_j\}) \in M^{\boldsymbol{\alpha}}_{\mathbf{P}^1}(\mathbf{t}, \boldsymbol{\lambda}, L) \,\middle|\, \dim \mathrm{End}(E, \{l^{(i)}_j\}) = 1 \right\}$$

is irreducible. There exists an integer n such that $h^1(E(n)) = 0$ for any $(E, \nabla, \{l^{(i)}_j\}) \in M^{\boldsymbol{\alpha}}_{\mathbf{P}^1}(\mathbf{t}, \boldsymbol{\lambda}, L)$. Then we can easily construct a smooth irreducible variety X and a flat family of parabolic bundles $(\tilde{E}, \{\tilde{l}^{(i)}_j\})$ on $\mathbf{P}^1 \times X$ over X such that the set of geometric fibers is just the set of all the simple

parabolic bundles $(E,\{l^{(i)}_j\})$ satisfying $h^1(E(n))=0$ and $\det E=L$. We can take an isomorphism $\det\tilde{E}\cong L\otimes\mathcal{L}$ for some line bundle $\mathcal{L}$ on X. The functor $\mathrm{Conn}_X:(Sch/X)\longrightarrow(Sets)$ defined by

$$\mathrm{Conn}_X(U)=\left\{\nabla:\tilde{E}_U\to\tilde{E}_U\otimes\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))\;\middle|\;\begin{array}{l}\nabla\text{ is a relative connection satisfying}\\ (\mathrm{res}_{t_i}(\nabla)-\lambda^{(i)}_j)((\tilde{l}^{(i)}_j)_U)\subset(\tilde{l}^{(i)}_{j+1})_U\text{ for any }i,j\\ \text{and }\det(\nabla)=(\nabla_L)_U\otimes\mathcal{L}\end{array}\right\}$$

is represented by a scheme Y of finite type over X. We can see by the same proof as Proposition 4.1 that Y is smooth over X whose fiber is an affine space of dimension $h^0((\mathcal{F}^1_0)_x)$ for $x\in X$, where

$$\mathcal{F}^1_0=\left\{s\in\mathcal{E}nd(\tilde{E})\otimes\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))\;\middle|\;\mathrm{Tr}(s)=0\text{ and }\mathrm{res}_{t_i\times X}(s)(\tilde{l}^{(i)}_j)\subset\tilde{l}^{(i)}_{j+1}\text{ for any }i,j\right\}.$$

So Y is irreducible. Let

$$\tilde{\nabla}:\tilde{E}_Y\longrightarrow\tilde{E}_Y\otimes\Omega^1_{\mathbf{P}^1}(t_1+\cdots+t_n)$$

be the universal relative connection and put

$$Y':=\left\{y\in Y\;\middle|\;(\tilde{E},\tilde{\nabla},\{\tilde{l}^{(i)}_j\})\otimes k(y)\text{ is }\boldsymbol{\alpha}\text{-stable}\right\}.$$

Then a morphism $Y'\to M^{\boldsymbol{\alpha}}_{\mathbf{P}^1}(\mathbf{t},\boldsymbol{\lambda},L)$ is induced whose image is just N. Thus N becomes irreducible.

In order to prove that $M^{\boldsymbol{\alpha}}_{\mathbf{P}^1}(\mathbf{t},\boldsymbol{\lambda},L)$ is irreducible, it suffices to show that $\dim(M^{\boldsymbol{\alpha}}_{\mathbf{P}^1}(\mathbf{t},\boldsymbol{\lambda},L)\setminus N)<(r-1)(rn-2r-2)$ because $M^{\boldsymbol{\alpha}}_{\mathbf{P}^1}(\mathbf{t},\boldsymbol{\lambda},L)$ is smooth of equidimension $(r-1)(rn-2r-2)$. Take any member $(E,\nabla,\{l^{(i)}_j\})\in M^{\boldsymbol{\alpha}}_{\mathbf{P}^1}(\mathbf{t},\boldsymbol{\lambda},L)\setminus N$. We can write

$$E\cong\mathcal{O}_{\mathbf{P}^1}(a_1)^{\oplus r_1}\oplus\mathcal{O}_{\mathbf{P}^1}(a_2)^{\oplus r_2}\oplus\cdots\oplus\mathcal{O}_{\mathbf{P}^1}(a_s)^{\oplus r_s}$$

with $a_1<a_2<\cdots<a_s$. If $l^{(k)}_{r-1}\not\subset\mathcal{O}_{\mathbf{P}^1}(a_2)^{\oplus r_2}|_{t_k}\oplus\cdots\oplus\mathcal{O}_{\mathbf{P}^1}(a_s)^{\oplus r_s}|_{t_k}$ for some k, then we replace $(E,\{l^{(i)}_j\})$ by its elementary transform along t_k by $l^{(k)}_{r-1}$, which is isomorphic to the bundle

$$\mathcal{O}_{\mathbf{P}^1}(a_1-1)^{\oplus r_1-1}\oplus\mathcal{O}_{\mathbf{P}^1}(a_1)\oplus\mathcal{O}_{\mathbf{P}^1}(a_2-1)^{\oplus r_2}\oplus\cdots\oplus\mathcal{O}_{\mathbf{P}^1}(a_s-1)^{\oplus r_s}$$

with a certain parabolic structure. Repeating this process, we finally obtain two cases:

$$\begin{cases}E\cong\mathcal{O}_{\mathbf{P}^1}(a)^{\oplus r}\\ l^{(i)}_{r-1}\subset\mathcal{O}_{\mathbf{P}^1}(a_2)^{\oplus r_2}|_{t_i}\oplus\cdots\oplus\mathcal{O}_{\mathbf{P}^1}(a_s)^{\oplus r_s}|_{t_i}\quad\text{for all }i\text{ and }r_1>0.\end{cases}$$

Consider the case $E\cong\mathcal{O}_{\mathbf{P}^1}(a)^{\oplus r}$. Tensoring $\mathcal{O}_{\mathbf{P}^1}(-a)$, we may assume $E\cong\mathcal{O}_{\mathbf{P}^1}^{\oplus r}$. For a suitable choice of a basis of $\mathcal{O}_{\mathbf{P}^1}^{\oplus r}$, we may assume that

$$l^{(1)}_{r-i}=k(t_1)e_1+k(t_1)e_2+\cdots+k(t_1)e_i$$

for $i=1,\dots,r$, where e_j is the vector of size r whose j-th component is 1 and others are zero. Then the group of automorphisms of E fixing $l^{(1)}_{r-1},l^{(1)}_{r-2},\dots,l^{(1)}_1$ is

$$B=\{(a_{ij})\in GL_r(\mathbf{C})\,|a_{ij}=0\text{ for }i>j\}.$$

We put

$$p(1):=\max\left\{p\;\middle|\;v_p\neq0\text{ for some }(v_1,\dots,v_r)\in l^{(2)}_{r-1}\right\}.$$

Applying a certain automorphism in B, we may assume that $l^{(2)}_{r-1}=k(t_2)e_{p(1)}$. Inductively we put

$$p(i):=\max\left\{p\;\middle|\;v_p\neq0\text{ for some }(v_1,\dots,v_r)\in l^{(2)}_{r-i}\text{ and }p\neq p(j)\text{ for any }j<i\right\}.$$

Applying an automorphism in B which fixes $l^{(2)}_{r-1},\dots,l^{(2)}_{r-i+1}$, we may assume that

$$l^{(2)}_{r-i}=l^{(2)}_{r-i+1}+k(t_2)e_{p(i)}.$$

Then the group of automorphisms of E fixing both $\{l^{(1)}_j\}$ and $\{l^{(2)}_j\}$ is

$$B'=\left\{(a_{ij})\in GL_r(\mathbf{C})\,\middle|a_{ij}=0\text{ for }i>j\text{ and }a_{p(i)p(j)}=0\text{ for }i>j\right\}.$$

Applying a certain automorphism in B', we may assume that $l^{(3)}_{r-1}$ can be generated by a vector $w = (w_1, \dots, w_r)$, where each w_i is either 1 or 0. Note that we have $n \geq 3$ by the assumption of the proposition. The group of automorphisms of E fixing $\{l^{(1)}_j\}$, $\{l^{(2)}_j\}$ and $l^{(3)}_{r-1}$ is

$$B'' = \left\{ (a_{ij}) \in GL_r(\mathbf{C}) \,\middle|\, \begin{array}{l} a_{ij}=0 \text{ for } i>j,\ a_{p(i)p(j)}=0 \text{ for } i>j \\ \text{and there is } c \in \mathbf{C}^{\times} \text{ satisfying } a_{ii}w_i + \sum a_{ik}w_k = cw_i \text{ for any } i \end{array} \right\}.$$

Since the parabolic bundle $(E, \{l^{(i)}_j\})$ has a nontrivial endomorphism, there is (i,j) with $i<j$ and $p(i)<p(j)$ or there is some i satisfying $w_i = 0$. Note that $r \geq 3$ or $n \geq 4$ because we assume that $rn-2r-2>0$. First assume that $r \geq 3$. If $w_{p(i)} = w_{p(j)} = 1$ and $p(i)<p(j)$ for some $i<j$, then we apply an automorphism of the form (a_{kl}) such that $a_{kk}=1$ if $k \neq p(i)$, $a_{kl}=0$ if $k \neq l$ and $(k,l) \neq (p(i),p(j))$, $a_{p(i)p(i)} = 1-c$ and $a_{p(i)p(j)} = c$ for some $c \in \mathbf{C} \setminus \{1\}$. Then we can parameterize $l^{(3)}_{r-2}$ modulo this action by an algebraic scheme whose dimension is less than $r-2$. If $w_i = 0$ for some i, then we apply an automorphism of the form (a_{kl}) such that $a_{kk}=1$ for $k \neq i$, $a_{ii} = c$ ($c \in \mathbf{C}^{\times}$) and $a_{kl}=0$ for $k \neq l$. Then $l^{(3)}_{r-2}$ can be parameterized modulo this action by an algebraic scheme of dimension less than $r-2$. Next assume that $n \geq 4$. Then we apply an automorphism as above for each case and $l^{(4)}_{r-1}$ can be parameterized by a variety of dimension less than $r-1$. Taking account of all the above arguments, we can parameterize parabolic bundles $(E, \{l^{(i)}_j\})$ satisfyng $E \cong \mathcal{O}_{\mathbf{P}^1}^{\oplus r}$ and $\dim \mathrm{End}(E, \{l^{(i)}_j\}) > 1$ by an algebraic scheme of dimension less than $(r-1)(rn/2-(r+1))$.

Consider the case when $E \cong \mathcal{O}_{\mathbf{P}^1}(a_1)^{\oplus r_1} \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(a_s)^{\oplus r_s}$ with $r_1>0$ and $l^{(i)}_{r-1} \subset \mathcal{O}_{\mathbf{P}^1}(a_2)^{\oplus r_2}|_{t_i} \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(a_s)^{\oplus r_s}|_{t_i}$ for any i. For each i, we consider an identification $\mathcal{O}_{\mathbf{P}^1}(a_i)^{\oplus r_i}|_{t_i} = \mathbf{C}^{r_i}$ and denote by $e^{(i)}_j$ the element of this vector space whose j-th component is 1 and others are zero. We put

$$p(1) := \min \left\{ p \,\middle|\, v_p \neq 0 \text{ for some } (v_1,\dots,v_s) \in l^{(1)}_{r-1} \text{ where } v_i \in \mathcal{O}_{\mathbf{P}^1}(a_i)^{\oplus r_i}|_{t_i} \text{ for any } i \right\}.$$

Applying an automorphism of E, we may assume that $l^{(1)}_{r-1}$ is generated by $e^{(p(1))}_{j(1)}$ with $j(1)=1$. Inductively we put

$$p(i) := \min \left\{ p \,\middle|\, v^{(p)}_j \neq 0 \text{ for some } \sum v^{(p)}_j e^{(p)}_j \in l^{(1)}_{r-i} \text{ where } (p,j) \neq (p(k),j(k)) \text{ for any } k<i \right\},$$

Applying an automorphism of E fixing $l^{(1)}_{r-1}, \dots, l^{(1)}_{r-i+1}$, we may assume that $l^{(1)}_{r-i} = l^{(1)}_{r-i+1} + k(t_1)e^{(p(i))}_{j(i)}$, where we put

$$j(i) := 1 + \max\left(\{0\} \cup \{j(i') | i'<i \text{ and } p(i') = p(i)\}\right).$$

Then each $l^{(1)}_{r-i}$ is generated by $e^{(p(1))}_{j(1)}, \dots, e^{(p(i))}_{j(i)}$ and the group of automorphisms of E fixing $\{l^{(1)}_j\}$ is

$$B = \left\{ (a^{pq}_{jk})^{1\leq p,q \leq s}_{1 \leq j \leq r_p, 1 \leq k \leq r_q} \,\middle|\, \begin{array}{l} (a^{pq}_{jk})_{1 \leq j \leq r_p, 1 \leq k \leq r_q} \in \mathrm{End}(\mathcal{O}(a_q)^{r_q}, \mathcal{O}(a_p)^{r_p}) \text{ for each } (p,q) \\ (a^{(pp)}_{jk}) \in \mathrm{Aut}(\mathcal{O}(a_p)^{r_p}) \text{ and } a^{p(i)p(i')}_{j(i)j(i')}(t_1) = 0 \text{ for } i>i' \end{array} \right\}.$$

Note that we have $j(i') < j(i)$ if $i'<i$ and $p(i') = p(i)$. We will also normalize $\{l^{(2)}_j\}$. First we put

$$(p'(1), -j'(1)) := \min \left\{ (p,-j) \,\middle|\, v^{(p)}_j \neq 0 \text{ for some } \sum v^{(p)}_j e^{(p)}_j \in l^{(2)}_{r-1} \right\},$$

where we consider lexicographic order on the pair $(p,-j)$. Applying an automorphism of E fixing $\{l^{(1)}_j\}$, we may assume that $l^{(2)}_{r-1}$ is generated by $e^{(p'(1))}_{j'(1)}$. Inductively we put

$$(p'(i), -j'(i)) := \min \left\{ (p,-j) \,\middle|\, \begin{array}{l} v^{(p)}_j \neq 0 \text{ for some } \sum v^{(p)}_j e^{(p)}_j \in l^{(2)}_{r-i} \text{ and} \\ (p,j) \neq (p'(k), j'(k)) \text{ for any } k<i \end{array} \right\}.$$

Applying an automorphism in B fixing $l^{(2)}_{r-1}, \dots, l^{(2)}_{r-i+1}$, we may assume that $l^{(2)}_{r-i} = l^{(2)}_{r-i+1} + k(t_2)e^{(p'(i))}_{j'(i)}$. Then the group of automorphisms of E fixing $\{l^{(1)}_j\}$ and $\{l^{(2)}_j\}$ is

$$B' = \left\{ (a^{pq}_{jk})^{1 \leq p,q \leq s}_{1 \leq j \leq r_p, 1 \leq k \leq r_q} \in B \,\middle|\, a^{p'(i)p'(l)}_{j'(i)j'(l)}(t_2) = 0 \text{ for } i>l \right\}.$$

Applying a certain automorphism in B', we may assume that $l^{(3)}_{r-1}$ can be generated by a vector $w = \sum w^{(p)}_j e^{(p)}_j$, where each $w^{(p)}_j$ is either 1 or 0 with respect to a certain identification $\mathcal{O}_{\mathbf{P}^1}(a_p)|_{t_3} \cong \mathbf{C}$. Then

the group of automorphisms of E fixing $\{l^{(1)}_j\}$, $\{l^{(2)}_j\}$ and $l^{(3)}_{r-1}$ is

$$B'' = \left\{ (a^{pq}_{jk})^{1\leq p,q\leq s}_{1\leq j\leq r_p, 1\leq k\leq r_q} \in B' \;\middle|\; \text{for some } c\in \mathbf{C}^{\times},\ \textstyle\sum_{q,k} a^{pq}_{jk} w^{(q)}_k = c w^{(p)}_j \text{ for any } p,j \right\}.$$

Note that $p(1)\geq 2$ because $l^{(1)}_{r-1}\subset \mathcal{O}_{\mathbf{P}^1}(a_2)^{\oplus r_2}|_{t_1}\oplus\cdots\oplus\mathcal{O}_{\mathbf{P}^1}(a_s)^{\oplus r_s}|_{t_1}$. If $w^{(1)}_1\neq 0$ and $w^{(p(1))}_{j(1)}\neq 0$, then there exist elements (a^{pq}_{jk}) of B'' satisfying $a^{p(1)1}_{j(1)1}(t_3)w^{(1)}_1 + a^{p(1)p(1)}_{j(1)j(1)} w^{(p(1))}_{j(1)} = w^{(p(1))}_{j(1)}$, $a^{pp}_{jj}=1$ for $(p,j)\neq (p(1),j(1))$ and $a^{(pq)}_{jk}=0$ for $(p,j)\neq (q,k)$ and $(p,j)\neq(p(1),j(1))$. Applying such automorphisms, $l^{(3)}_{r-2}$ can be parameterized modulo this action by an algebraic scheme of dimension less than $r-2$ when $r\geq 3$ and $l^{(4)}_{r-1}$ can be parametrized modulo this action by an algebraic scheme of dimension less than $r-1$ when $n\geq 4$. If $w^{(p_0)}_{j_0}=0$ for some (p_0,j_0), then we apply an automorphism (a^{pq}_{jk}) in B'' satisfying $a^{pp}_{jj}=1$ for $(p,j)\neq (p_0,j_0)$, $a^{p_0p_0}_{j_0j_0}=c$ ($c\in\mathbf{C}^{\times}$) and $a^{pq}_{jk}=0$ for $(p,j)\neq(q,k)$. Then $l^{(3)}_{r-2}$ can be parameterized modulo this action by an algebraic scheme of dimension less than $r-2$ when $r\geq 3$ and $l^{(4)}_{r-1}$ can be parameterized modulo this action by an algebraic scheme of dimension less than $r-1$ when $n\geq 4$. Therefore we can parameterize parabolic bundles $(E,\{l^{(i)}_j\})$ satisfying $E\cong\mathcal{O}_{\mathbf{P}^1}(a_1)^{\oplus r_1}\oplus\cdots\oplus\mathcal{O}_{\mathbf{P}^1}(a_s)^{\oplus r_s}$ and $l^{(i)}_{r-1}\subset\mathcal{O}_{\mathbf{P}^1}(a_2)^{\oplus r_2}|_{t_i}\oplus\cdots\oplus\mathcal{O}_{\mathbf{P}^1}(a_s)^{\oplus r_s}|_{t_i}$ for any i by an algebraic scheme of dimension less than $(r-1)(rn/2-r-1)$.

From all the above arguments, we can construct an algebraic scheme Z of dimension less than $(r-1)(rn/2-r-1)$ and a flat family $(\tilde{E},\{\tilde{l}^{(i)}_j\})$ of parabolic bundles on $\mathbf{P}^1_Z$ over Z such that the set of the geometric fibers of $(\tilde{E},\{\tilde{l}^{(i)}_j\})$ is just the set of parabolic bundles $(E,\{l^{(i)}_j\})$ satisfying $\operatorname{rank} E=r$, $\deg E=d$, $\dim\operatorname{End}(E,\{l^{(i)}_j\})\geq 2$ and $h^1(E(n))=0$. Note that the functor $\mathrm{Conn}_Z : (Sch/Z)\longrightarrow (Sets)$ defined by

$$\mathrm{Conn}_Z(S) = \left\{ \nabla : \tilde{E}_S\to\tilde{E}_S\otimes\Omega^1_{\mathbf{P}^1}(t_1+\cdots+t_n) \;\middle|\; \begin{array}{l} \nabla \text{ is a relative connection satisfying} \\ (\mathsf{res}_{t_i\times S}(\nabla)-\lambda^{(i)}_j)(\tilde{l}^{(i)}_j)_S\subset(\tilde{l}^{(i)}_{j+1})_S \text{ for any } i,j \\ \text{and } \det(\nabla)=(\nabla_L)_S\otimes\mathcal{L} \end{array} \right\}$$

can be represented by a scheme W of finite type over Z. Let $\tilde{\nabla}:\tilde{E}_W\to\tilde{E}_W\otimes\Omega^1_{\mathbf{P}^1}(t_1+\cdots+t_n)$ be the universal relative connection. Then there is a stratification

$$W = \coprod_{m>(r-1)(rn/2-r-1)} W_m,$$

where

$$W_m = \left\{ p\in W \;\middle|\; \dim H^0(\mathcal{F}^1\otimes k(p)) = m \right\}.$$

Here we put

$$\mathcal{F}^1 = \left\{ s\in \mathcal{E}nd(\tilde{E})\otimes\Omega^1_{\mathbf{P}^1}(t_1+\cdots+t_n) \;\middle|\; \mathrm{Tr}(s)=0 \text{ and } \mathsf{res}_{t_i\times Z}(s)(\tilde{l}^{(i)}_j)\subset\tilde{l}^{(i)}_{j+1} \text{ for any } i,j \right\}.$$

Then the fiber of W_m over $p\in Z$ is isomorphic to $H^0(\mathcal{F}^1\otimes k(p))\cong\mathbf{C}^m$ if it is non-empty. If we put

$$W'_m := \left\{ p\in W_m \;\middle|\; (\tilde{E},\tilde{\nabla},\{\tilde{l}^{(i)}_j\})\otimes k(p) \text{ is } \boldsymbol{\alpha}\text{-stable} \right\},$$

then a morphism $f : W'_m\to M^{\boldsymbol{\alpha}}_{\mathbf{P}^1}(\mathbf{t},\boldsymbol{\lambda},L)$ is induced. Note that we have to show that $\dim f(W'_m) < (r-1)(rn-2r-2)$ in order to prove the inequality $\dim(M^{\boldsymbol{\alpha}}_{\mathbf{P}^1}(\mathbf{t},\boldsymbol{\lambda},L)\setminus N) < (r-1)(rn-2r-2)$. Since $\dim W'_m\leq m+\dim Z < m+(r-1)(rn/2-r-1)$, it suffices to show that $\dim f^{-1}(q)\geq m-(r-1)(rn/2-r-1)$ for every $q\in f(W'_m)$. Let $(E,\nabla,\{l^{(i)}_j\})$ be the parabolic connection corresponding to q. Take any point $z\in Z$ satisfying $(\tilde{E},\{\tilde{l}^{(i)}_j\})\otimes k(z)\cong (E,\{l^{(i)}_j\})$. Note that the fiber $(W_m)_z$ of W'_m over z is isomorphic to the affine space isomorphic to $H^0(\mathcal{F}^1\otimes k(z))\cong\mathbf{C}^m$. We will show that $\dim((W'_m)_z\cap f^{-1}(q))\geq m-(r-1)(rn/2-r-1)$. There is a canonical action of the group $\mathrm{Aut}(E,\{l^{(i)}_j\})$ on the fiber $\dim(W'_m)_z$ and we have $f(gp)=p$ for any $p\in (W'_m)_z\cap f^{-1}(q)$ and $g\in\mathrm{Aut}(E,\{l^{(i)}_j\})$. Since the tangent map of the orbit map $\mathrm{Aut}((\tilde{E},\{\tilde{l}^{(i)}_j\})\otimes k(z))\ni g\mapsto gp\in (W'_m)_z$ is the linear map

$$H^0(\mathcal{F}^0\otimes k(p))\mapsto H^0(\mathcal{F}^1\otimes k(p)) : \quad s\mapsto\tilde{\nabla}s-s\tilde{\nabla}$$

which is injective by the stability of $(\tilde{E},\tilde{\nabla},\{\tilde{l}^{(i)}_j\})\otimes k(p)$, we have

$$\begin{aligned}\dim((W_m)_z\cap f^{-1}(q)) &\geq \dim H^0(\mathcal{F}^0\otimes k(p)) \\ &= \dim H^0(\mathcal{F}^1\otimes k(p))-(r-1)(rn/2-r-1) \\ &= m-(r-1)(rn/2-r-1).\end{aligned}$$

Here we put

$$\mathcal{F}^0=\left\{ s\in\mathcal{E}nd(\tilde{E}) \,\middle|\, \mathrm{Tr}(s)=0 \text{ and } s|_{t_i\times Z}(\tilde{l}^{(i)}_j)\subset\tilde{l}^{(i)}_j \text{ for any } i,j\right\}.$$

So we obtain the desired inequality $\dim f^{-1}(q)\geq m-(r-1)(rn/2-r-1)$. □

5. Riemann-Hilbert correspondence

We take a point $x\in T$ and denote $\mathcal{C}_x$ and $\tilde{\mathbf{t}}_x$ simply by C and $\mathbf{t}$. We also denote the fiber of $M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)$ over $x\in T$ by $M^{\boldsymbol{\alpha}}_C(\mathbf{t},r,d)$. Sending $(E,\nabla,\{l^{(i)}_*\}_{1\leq i\leq n})$ to $\ker\nabla^{an}|_{C\setminus\{t_1,\ldots,t_n\}}$, we obtain a holomorphic mapping

$$\mathbf{RH}_x: M^{\boldsymbol{\alpha}}_C(\mathbf{t},r,d)\longrightarrow RP_r(C,\mathbf{t}),$$

which makes the diagram

$$\begin{array}{ccc} M^{\boldsymbol{\alpha}}_C(\mathbf{t},r,d) & \xrightarrow{\mathbf{RH}_x} & RP_r(C,\mathbf{t}) \\ \downarrow & & \downarrow \\ \Lambda^{(n)}_r(d) & \xrightarrow{rh} & \mathcal{A}^{(n)}_r \end{array}$$

commute. Here $rh:\Lambda^{(n)}_r(d)\ni\boldsymbol{\lambda}=(\lambda^{(i)}_j)\mapsto\mathbf{a}=(a^{(i)}_j)\in\mathcal{A}^{(n)}_r$ is defined by

$$\prod_{j=0}^{r-1}(X-\exp(-2\pi\sqrt{-1}\lambda^{(i)}_j))=X^r+a^{(i)}_{r-1}X^{r-1}+\cdots+a^{(i)}_1X+a^{(i)}_0$$

for $i=1,\ldots,n$. If we denote the fiber of $M^{\boldsymbol{\alpha}}_C(\mathbf{t},r,d)$ over $\boldsymbol{\lambda}\in\Lambda^{(n)}_r(d)$ by $M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda})$, then $\mathbf{RH}_x$ induces a holomorphic mapping

$$\mathbf{RH}_{(x,\boldsymbol{\lambda})}: M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda})\longrightarrow RP_r(C,\mathbf{t})_{\mathbf{a}}$$

for $rh(\boldsymbol{\lambda})=\mathbf{a}$. In this section we will prove the properness of this morphism $\mathbf{RH}_{(x,\boldsymbol{\lambda})}$ and obtain Theorem 2.2. In order to prove the properness of $\mathbf{RH}_{(x,\boldsymbol{\lambda})}$, it is essential to prove the surjectivity of $\mathbf{RH}_{(x,\boldsymbol{\lambda})}$ and compactness of every fiber of $\mathbf{RH}_{(x,\boldsymbol{\lambda})}$. For the proof of the surjectivity of $\mathbf{RH}_{(x,\boldsymbol{\lambda})}$, it is essential to use the following Langton's type theorem:

Lemma 5.1. *Let R be a discrete valuation ring with residue field $k=R/m$ and quotient field K. Take a flat family $(E,\nabla,\{l^{(i)}_j\})$ of parabolic connections on $C\times\operatorname{Spec}R$ over $\operatorname{Spec}R$ whose generic fiber $(E,\nabla,\{l^{(i)}_j\})\otimes K$ is $\boldsymbol{\alpha}$-semistable. Then we can obtain, by repeating elementary transformations of $(E,\nabla,\{l^{(i)}_j\})$ along the special fiber $C\times\operatorname{Spec}k$, a flat family $(E',\nabla',\{l'^{(i)}_j\})$ of parabolic connections on $C\times\operatorname{Spec}R$ over $\operatorname{Spec}R$ such that $(E',\nabla',\{l'^{(i)}_j\})\otimes K\cong(E,\nabla,\{l^{(i)}_j\})\otimes K$ and $(E',\nabla',\{l'^{(i)}_j\})\otimes k$ is $\boldsymbol{\alpha}$-semistable.*

Proof. This is just a corollary of the proof of [[7], Proposition 5.5]. □

Proposition 5.1. *The morphism $\mathbf{RH}_{(x,\boldsymbol{\lambda})}: M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda})\to RP_r(C,\mathbf{t})_{\mathbf{a}}$ is surjective for any $\boldsymbol{\lambda}\in\Lambda^{(n)}_r(d)$ satisfying $rh(\boldsymbol{\lambda})=\mathbf{a}$.*

Proof. By Proposition 3.1, we obtain an isomorphism

$$\sigma:\mathcal{M}_C(\mathbf{t},\boldsymbol{\lambda})\xrightarrow{\sim}\mathcal{M}_C(\mathbf{t},\boldsymbol{\mu}) \tag{8}$$

of moduli stacks of parabolic connections, where $0\leq\mathrm{Re}(\mu^{(i)}_j)<1$ for any i,j. Note that σ induces an isomorphism

$$M^{\mathrm{irr}}_C(\mathbf{t},\boldsymbol{\lambda})\xrightarrow[\sim]{\sigma}M^{\mathrm{irr}}_C(\mathbf{t},\boldsymbol{\mu}),$$

where $M^{\mathrm{irr}}_C(\mathbf{t},\boldsymbol{\lambda})$ (resp. $M^{\mathrm{irr}}_C(\mathbf{t},\boldsymbol{\mu})$ is the moduli space of irreducible $(\mathbf{t},\boldsymbol{\lambda})$-parabolic connections (resp. $(\mathbf{t},\boldsymbol{\mu})$-parabolic connections). By construction, we have $\mathbf{RH}_{(x,\boldsymbol{\mu})}\circ\sigma|_{M^{\mathrm{irr}}_C(\mathbf{t},\boldsymbol{\lambda})}=\mathbf{RH}_{(x,\boldsymbol{\lambda})}|_{M^{\mathrm{irr}}_C(\mathbf{t},\boldsymbol{\lambda})}$.

Consider the restriction

$$\mathbf{RH}_{(x,\boldsymbol{\lambda})} : M_C^{\mathrm{irr}}(\mathbf{t},\boldsymbol{\lambda}) \longrightarrow RP_r(C,\mathbf{t})_{\mathbf{a}}^{\mathrm{irr}},$$

of $\mathbf{RH}$, where $M_C^{\mathrm{irr}}(\mathbf{t},\boldsymbol{\lambda})$ is the moduli space of irreducible parabolic connections and $RP_r(C,\mathbf{t})_{\mathbf{a}}^{\mathrm{irr}}$ is the moduli space of irreducible representations. For any point $\rho \in RP_r(C,\mathbf{t})_{\mathbf{a}}^{\mathrm{irr}}$, ρ corresponds to a holomorphic vector bundle E° on $C \setminus \{t_1,\dots,t_n\}$ and a holomorphic connection $\nabla^\circ : E^\circ \to E^\circ \otimes \Omega^1_{C\setminus\{t_1,\dots,t_n\}}$. By [[4], Proposition 5.4], there is a unique pair (E,∇) of a holomorphic vector bundle E on C and a logarithmic connection $\nabla : E \to E\otimes\Omega^1_C(t_1+\cdots+t_n)$ such that $(E,\nabla)|_{C\setminus\{t_1,\dots,t_n\}} \cong (E^\circ,\nabla^\circ)$ and that all the eigenvalues of $\mathsf{res}_{t_i}(\nabla)$ lies in $\{z\in\mathbf{C} | 0 \leq \mathrm{Re}(z) < 1\}$. We can take a parabolic structure $\{l^{(i)}_j\}$ on E which is compatible with the connction ∇. Then $(E,\nabla,\{l^{(i)}_j\})$ becomes a $(\mathbf{t},\boldsymbol{\mu})$-parabolic connection such that $\mathbf{RH}_{(x,\boldsymbol{\mu})}(E,\nabla,\{l^{(i)}_j\}) = \rho$. Thus we have $\sigma^{-1}(E,\nabla,\{l^{(i)}_j\}) \in M_C^{\mathrm{irr}}(\mathbf{t},\boldsymbol{\lambda}) \subset M_C^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda})$ (note that an irreducible parabolic connection is automatically $\boldsymbol{\alpha}$-stable) and $\mathbf{RH}_{(x,\boldsymbol{\lambda})}(\sigma^{-1}(E,\nabla,\{l^{(i)}_j\})) = \rho$.

So it is sufficient to show that any member $[\rho] \in RP_r(C,\mathbf{t})_{\mathbf{a}}$ corresponding to a reducible representation is contained in the image of $\mathbf{RH}_{(x,\boldsymbol{\lambda})}$, in order to prove the surjectivity of $\mathbf{RH}_{(x,\boldsymbol{\lambda})}$. The fundamental group $\pi_1(C\setminus\{t_1,\dots,t_n\},*)$ is generated by cycles $\alpha_1,\beta_1,\dots,\alpha_g,\beta_g$ and loops γ_i around t_i for $i=1,\dots,n$ whose relation is given by

$$\prod_{i=1}^{g} \alpha_i\beta_i\alpha_i^{-1}\beta_i^{-1} \prod_{j=1}^{n} \gamma_j = 1.$$

Since $n>0$, the fundamental group is isomorphic to the free group generated by the free generators $\alpha_1,\beta_1,\dots,\alpha_g,\beta_g,\gamma_1,\dots,\gamma_{n-1}$. So the space $\mathrm{Hom}(\pi_1(C\setminus\{t_1,\dots,t_n\},*),GL_r(\mathbf{C}))$ of representations is isomorphic to $GL_r(\mathbf{C})^{2g+n-1}$ and it is irreducible. Thus there exist a smooth affine curve T and a morphism $f : T \to \mathrm{Hom}(\pi_1(C\setminus\{t_1,\dots,t_n\},*),GL_r(\mathbf{C}))$ such that the image $f(p_0)$ of a special point $p_0\in T$ corresponds to the representation ρ and the image $f(\eta)$ of a generic point $\eta\in T$ corresponds to an irreducible representation. Since the morphism f corresponds to a family of representations of the fundamental group $\pi_1(C\setminus\{t_1,\dots,t_n\},*)$, we can construct a holomorphic vector bundle $\tilde{E}^\circ$ on $(C\setminus\{t_1,\dots,t_n\})\times T$ and a holomorphic connection $\tilde{\nabla}^\circ : \tilde{E}^\circ \to \tilde{E}^\circ\otimes\Omega^1_{(C\setminus\{t_1,\dots,t_n\})\times T/T}$ such that $(\tilde{E}^\circ,\tilde{\nabla}^\circ)|_{(C\setminus\{t_1,\dots,t_n\})\times\eta}$ corresponds to the representation $f(\eta)$ which is irreducible and $(\tilde{E}^\circ,\tilde{\nabla}^\circ)|_{(C\setminus\{t_1,\dots,t_n\})\times p_0}$ corresponds to the representation ρ. By the relative version of [[4], Proposition 5.4], we can construct, after replacing T by a neighborhood of p_0 if necessary, a holomorphic vector bundle $\tilde{E}$ on $C\times T$ and a relative connection $\tilde{\nabla} : \tilde{E} \to \tilde{E}\otimes\Omega^1_C(t_1+\cdots+t_n)$ such that $(\tilde{E},\tilde{\nabla})|_{(C\setminus\{t_1,\dots,t_n\})\times T} \cong (\tilde{E}^\circ,\tilde{\nabla}^\circ)$ and that all the eigenvalues of $\mathsf{res}_{t_i}(\tilde{\nabla}\otimes k(t))$ lies in $\{z\in\mathbf{C} | -\epsilon < \mathrm{Re}(z) < 1-\epsilon\}$ for any $t\in T$, where $\epsilon>0$ depends on the residue of $\tilde{\nabla}\otimes k(p_0)$. We can take a flat family of parabolic structures $\{\tilde{l}^{(i)}_j\}$ on $\tilde{E}$ which is compatible with $\tilde{\nabla}$ and $(\tilde{E},\tilde{\nabla},\{\tilde{l}^{(i)}_j\})$ becomes a flat family of parabolic connections over T. Note that the transform σ in (**??**) canonically extends to a transform $\tilde{\sigma} : \mathcal{M}_1 \to \mathcal{M}_2$, where $\mathcal{M}_1$ is a neighborhood of $\mathcal{M}_C(\mathbf{t},\boldsymbol{\lambda})$ in the moduli stack $\mathcal{M}_C(\mathbf{t},r,d)$ of parabolic connections and $\mathcal{M}_2$ is a neighborhood of $\mathcal{M}_C(\mathbf{t},\boldsymbol{\mu})$ in $\mathcal{M}_C(\mathbf{t},r,d')$ for some d'. Then we obtain a flat family $\tilde{\sigma}^{-1}(\tilde{E},\tilde{\nabla},\{\tilde{l}^{(i)}_j\})$ of parabolic connections, whose restriction to $C\times\eta$ is irreducible for generic $\eta\in T$. By Lemma 5.1, we can obtain, after replacing T by a neighborhood of p_0, a flat family $(\tilde{E}',\tilde{\nabla}',\{\tilde{l}'^{(i)}_j\})$ of $\boldsymbol{\alpha}$-semistable parabolic connections over T such that $(\tilde{E}',\tilde{\nabla}',\{\tilde{l}'^{(i)}_j\})|_{C\times(T\setminus\{p_0\})} \cong \tilde{\sigma}^{-1}(\tilde{E},\tilde{\nabla},\{\tilde{l}^{(i)}_j\})|_{C\times(T\setminus\{p_0\})}$. Since we take $\boldsymbol{\alpha}$ so generic that $\boldsymbol{\alpha}$-semistable $\Leftrightarrow$ $\boldsymbol{\alpha}$-stable, this family determines a morphism $g : T \to M_C^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda},L)$. By construction, we have $\mathbf{RH}_{(x,\boldsymbol{\lambda})}(g(p_0)) = [\rho]$. Hence $\mathbf{RH}_{(x,\boldsymbol{\lambda})}$ is surjective. □

Remark 5.1. It was not obvious in Proposition 4.1, 4.2 and 4.3 that the moduli space $M_C^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda},L)$ is non-empty. However, we can see now that it is in fact non-empty in the following way: We first take a simple parabolic bundle $(E,\{l^{(i)}_j\})$ of rank r on C such that $\det E\cong L$. For the existence of such a parabolic bundle, we may assume, by taking cetain elementary transform, that r and $\deg L$ are coprime. Then the existence for $g\geq 1$ follows because the moduli space of stable bundles of rank r with the fixed determinant L on a curve of genus $g\geq 1$ is non-empty. For the case of $C=\mathbf{P}^1$, we may assume that $E=\mathcal{O}_{\mathbf{P}^1}^{\oplus r}$ and we can take a parabolic structure $\{l^{(i)}_j\}$ on E such that $\mathrm{End}(E,\{l^{(i)}_j\})\cong\mathbf{C}$ because $n\geq 3$. We want to construct a connection $\nabla : E\to E\otimes\Omega^1_C(t_1+\cdots+t_n)$ such that $(E,\nabla,\{l^{(i)}_j\})$ becomes a $(\mathbf{t},\boldsymbol{\lambda})$-parabolic connection with the determinant (L,∇_L). The obstruction class for the existence of such

a connection lies in $H^1(\mathcal{F}_0^1)$, where

$$\mathcal{F}_0^1 = \left\{ s \in \mathcal{E}nd(E) \otimes \Omega_C^1(t_1 + \cdots + t_n) \,\middle|\, \mathrm{Tr}(s) = 0 \text{ and } \mathsf{res}_{t_i}(a)(l_j^{(i)}) \subset l_{j+1}^{(i)} \text{ for any } i, j \right\}.$$

If we put

$$\mathcal{F}_0^0 := \left\{ s \in \mathcal{E}nd(E) \,\middle|\, \mathrm{Tr}(s) = 0 \text{ and } s|_{t_i}(l_j^{(i)}) \subset l_j^{(i)} \text{ for any } i, j \right\},$$

then we have an isomorphism $H^1(\mathcal{F}_0^1) \cong H^0(\mathcal{F}_0^0)^\vee$. Since $\mathrm{End}(E, \{l_j^{(i)}\}) \cong \mathbf{C}$, we have $H^0(\mathcal{F}_0^0) = 0$ and so $H^1(\mathcal{F}_0^1) = 0$. Thus there is a connection $\nabla : E \to E \otimes \Omega_C^1(t_1 + \cdots + t_n)$ such that $(E, \nabla, \{l_j^{(i)}\})$ becomes a $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connection. If this connection is irreducible, then $(E, \nabla, \{l_j^{(i)}\})$ becomes $\boldsymbol{\alpha}$-stable and the moduli space $M_C^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ becomes non-empty. If $(E, \nabla, \{l_j^{(i)}\})$ is reducible, then we take the representation ρ of $\pi_1(C \setminus \{t_1, \ldots, t_n\}, *)$ which corresponds to $\ker(\nabla^{an}|_{C \setminus \{t_1, \ldots, t_n\}})$. $[\rho]$ determines a point of $RP_r(C, \mathbf{t})_{\mathbf{a}}$, where $rh(\boldsymbol{\lambda}) = \mathbf{a}$. Since $\mathbf{RH}_{(x,\boldsymbol{\lambda})}$ is surjective, we can take an $\boldsymbol{\alpha}$-stable $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connection $(E', \nabla', \{l'^{(i)}_j\})$ with the determinant (L, ∇_L) such that $\mathbf{RH}_{(x,\boldsymbol{\lambda})}(E', \nabla', \{l'^{(i)}_j\}) = [\rho]$. Hence we have $M_C^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L) \neq \emptyset$.

Proposition 5.2. *Every fiber of $\mathbf{RH}_{(x,\boldsymbol{\lambda})} : M_C^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}) \to RP_r(C, \mathbf{t})_{\mathbf{a}}$ is compact.*

Proof. Take any irreducible representation $\rho \in RP_r(C, \mathbf{t})_{\mathbf{a}}^{\mathrm{irr}}$. We want to show that the fiber $\mathbf{RH}^{-1}_{(x,\boldsymbol{\lambda})}(\rho)$ in $M_C^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda})$ is compact. By Proposition 3.1, we can obtain an isomorphism

$$\sigma : \mathcal{M}_C(\mathbf{t}, \boldsymbol{\lambda}) \xrightarrow{\sim} \mathcal{M}_C(\mathbf{t}, \boldsymbol{\mu}),$$

where $\mathcal{M}_C(\mathbf{t}, \boldsymbol{\lambda})$ (resp. $\mathcal{M}_C(\mathbf{t}, \boldsymbol{\mu})$) is the moduli stack of $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connections (resp. $(\mathbf{t}, \boldsymbol{\mu})$-parabolic connections)) without stability condition and $0 \leq \mathrm{Re}(\mu_j^{(i)}) < 1$ for any i, j. Then σ induces an isomorphism

$$\sigma : M_C^{\mathrm{irr}}(\mathbf{t}, \boldsymbol{\lambda}) \xrightarrow{\sim} M_C^{\mathrm{irr}}(\mathbf{t}, \boldsymbol{\mu})$$

between the moduli spaces of irreducible parabolic connections. Note that $M_C^{\mathrm{irr}}(\mathbf{t}, \boldsymbol{\lambda}) \subset M_C^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda})$. So σ also induces an isomorphism

$$\mathbf{RH}^{-1}_{(x,\boldsymbol{\lambda})}(\rho) \xrightarrow{\sim} \mathbf{RH}^{-1}_{(x,\boldsymbol{\mu})}(\rho).$$

By [[4], Proposition 5.4], there is a unique pair (E, ∇_E) of a bundle E on C and a logarithmic connection $\nabla_E : E \to E \otimes \Omega_C^1(t_1 + \cdots + t_n)$ such that the local system $\ker(\nabla_E^{an})|_{C \setminus \{t_1, \ldots, t_n\}}$ corresponds to the representation ρ and all the eigenvalues of $\mathsf{res}_{t_i}(\nabla_E)$ lies in $\{z \in \mathbf{C} | 0 \leq \mathrm{Re}(z) < 1\}$. Then the fiber $\mathbf{RH}^{-1}_{(x,\boldsymbol{\mu})}(\rho)$ is just the moduli space of parabolic structures $\{l_j^{(i)}\}$ on E which satisfy $(\mathsf{res}_{t_i}(\nabla_E) - \mu_j^{(i)})(l_j^{(i)}) \subset l_{j+1}^{(i)}$ for any i, j. So $\mathbf{RH}^{-1}_{(x,\boldsymbol{\mu})}(\rho)$ becomes a Zariski closed subset of a product of flag varieties. Thus $\mathbf{RH}^{-1}_{(x,\boldsymbol{\mu})}(\rho)$ is compact. Since the fiber $\mathbf{RH}^{-1}_{(x,\boldsymbol{\lambda})}(\rho)$ of $\mathbf{RH}_{(x,\boldsymbol{\lambda})}$ over ρ in $M_C^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda})$ is isomorphic to $\mathbf{RH}^{-1}_{(x,\boldsymbol{\mu})}(\rho)$ via σ, it is also compact.

Now take a reducible representation $[\rho] \in RP_r(C, \mathbf{t})_{\mathbf{a}}$. We may assume $\rho = \rho_1 \oplus \cdots \oplus \rho_s$ with each ρ_p irreducible. We can see that $\mathbf{RH}^{-1}_{(x,\boldsymbol{\lambda})}([\rho])$ is a constructible subset of $M_C^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda})$. So it suffices to show that $\mathbf{RH}^{-1}_{(x,\boldsymbol{\lambda})}([\rho])$ is stable under specialization in the compact moduli space $\overline{M_C^{D(\mathbf{t}), \boldsymbol{\alpha}', \boldsymbol{\beta}, \gamma}}(r, d, \{1\}_{1 \leq i \leq nr})$, in order to prove the compactness of $\mathbf{RH}^{-1}_{(x,\boldsymbol{\lambda})}([\rho])$. Here $\overline{M_C^{D(\mathbf{t}), \boldsymbol{\alpha}', \boldsymbol{\beta}, \gamma}}(r, d, \{1\}_{1 \leq i \leq nr})$ is the moduli scheme of $(\boldsymbol{\alpha}', \boldsymbol{\beta}, \gamma)$-stable parabolic $\Lambda^1_{D(\mathbf{t})}$-triples appeared in Theorem 3.1, which contains $M_C^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda})$ as a locally closed subscheme. Take any scheme point $x_1 \in \mathbf{RH}^{-1}_{(x,\boldsymbol{\lambda})}([\rho])$ and a point x_0 of the closure $\overline{\{x_1\}}$ of $\{x_1\}$ in $\overline{M_C^{D(\mathbf{t}), \boldsymbol{\alpha}', \boldsymbol{\beta}, \gamma}}(r, d, \{1\}_{1 \leq i \leq nr})$. Let K be the residue field $k(x_1)$ of x_1. Then there exists a discrete valuation ring R with quotient field K which dominates $\mathcal{O}_{x_0}$. The inclusion $\mathcal{O}_{x_0} \hookrightarrow R$ induces a flat family $(\tilde{E}_1, \tilde{E}_2, \tilde{\Phi}, F_*(\tilde{E}_1))$ of parabolic $\Lambda^1_{D(\mathbf{t})}$-triples on $C \times \operatorname{Spec} R$ over R. Here the left $\mathcal{O}_{C \times \operatorname{Spec} R}$-homomorphism $\tilde{\Phi} : \Lambda^1_{D(\mathbf{t})} \otimes \tilde{E}_1 \to \tilde{E}_2$ corresponds to a pair $(\tilde{\phi}, \tilde{\nabla})$ of an $\mathcal{O}_{C \times \operatorname{Spec} R}$-homomorphism $\tilde{\phi} : \tilde{E}_1 \to \tilde{E}_2$ and a morphism $\tilde{\nabla} : \tilde{E}_1 \to \tilde{E}_2 \otimes \Omega_C^1(t_1 + \cdots + t_n)$ satisfying $\tilde{\nabla}(as) = \phi(s) \otimes da + a\tilde{\nabla}(s)$ for $a \in \mathcal{O}_{C \times \operatorname{Spec} R}$ and $s \in \tilde{E}_1$. We also denote $(\tilde{E}_1, \tilde{E}_2, \tilde{\Phi}, F_*(\tilde{E}_1))$ by $(\tilde{E}_1, \tilde{E}_2, \tilde{\phi}, \tilde{\nabla}, F_*(\tilde{E}_1))$. Let η be the generic point of $\operatorname{Spec} R$ and ξ be the closed point of $\operatorname{Spec} R$ and put

$$(E_1, E_2, \phi, \nabla, F_*(E_1)) := (\tilde{E}_1, \tilde{E}_2, \tilde{\phi}, \tilde{\nabla}, F_*(\tilde{E}_1)) \otimes K$$

$$(E'_1, E'_2, \phi', \nabla', F_*(E'_1)) := (\tilde{E}_1, \tilde{E}_2, \tilde{\phi}, \tilde{\nabla}, F_*(\tilde{E}_1)) \otimes k(\xi).$$

Note that ϕ is isomorphic and $\nabla_E = (\phi)^{-1}\circ\nabla$ becomes a connection on $E := E_1$. There exists a filtration

$$(0,0) = (F^{(0)}, \nabla^{(0)}) \subset (F^{(1)}, \nabla^{(1)}) \subset \cdots \subset (F^{(s)}, \nabla^{(s)}) = (E, \nabla_E)$$

such that each $(F^{(p)}/F^{(p-1)}, \overline{\nabla^{(p)}})$ is irreducible, where $\overline{\nabla^{(p)}}$ is the connection on $F^{(p)}/F^{(p+1)}$ induced by $\nabla^{(p)}$. We may assume that $(F^{(p)}/F^{(p-1)}, \overline{\nabla^{(p)}})$ corresponds to the representation ρ_p. The filtration $\{(F^{(p)}, \nabla^{(p)})\}$ induces filtrations

$$0 = \tilde{F}_1^{(0)} \subset \tilde{F}_1^{(1)} \subset \cdots \subset \tilde{F}_1^{(s)} = \tilde{E}_1$$
$$0 = \tilde{F}_2^{(0)} \subset \tilde{F}_2^{(1)} \subset \cdots \subset \tilde{F}_2^{(s)} = \tilde{E}_2$$

such that each $\tilde{F}_j^{(p)}/\tilde{F}_j^{(p-1)}$ is flat over R for $j = 1,2$ and $\tilde{F}_1^{(p)} \otimes K = F^{(p)}$, $\tilde{F}_2^{(p)} \otimes K = \phi(F^{(p)})$. By construction, we have $\tilde{\phi}(\tilde{F}_1^{(p)}) \subset \tilde{F}_2^{(p)}$ and $\tilde{\nabla}(\tilde{F}_1^{(p)}) \subset \tilde{F}_2^{(p)} \otimes \Omega_C^1(t_1 + \cdots + t_n)$. Let

$$\overline{\tilde{\phi}^{(p)}} : \tilde{F}_1^{(p)}/\tilde{F}_1^{(p-1)} \longrightarrow \tilde{F}_2^{(p)}/\tilde{F}_2^{(p-1)}$$
$$\overline{\tilde{\nabla}^{(p)}} : \tilde{F}_1^{(p)}/\tilde{F}_1^{(p-1)} \longrightarrow \tilde{F}_2^{(p)}/\tilde{F}_2^{(p-1)} \otimes \Omega_C^1(t_1 + \cdots + t_n)$$

be the morphisms induced by $\tilde{\phi}$ and $\tilde{\nabla}$. We can construct a parabolic structure on $(F^{(p)}/F^{(p-1)}, \overline{\nabla^{(p)}})$ and obtain a $(\mathbf{t}, \boldsymbol{\lambda}^{(p)})$-parabolic connections of rank r_p and degree d_p. Extending this parabolic structure, we can obtain a flat family parabolic structures $F_*(\tilde{F}_1^{(p)}/\tilde{F}_1^{(p-1)})$ on $\tilde{F}_1^{(p)}/\tilde{F}_1^{(p-1)}$. Repeating elementary transforms of the flat family

$$(\tilde{F}_1^{(p)}/\tilde{F}_1^{(p-1)}, \tilde{F}_2^{(p)}/\tilde{F}_2^{(p-1)}, \overline{\tilde{\phi}^{(p)}}, \overline{\tilde{\nabla}^{(p)}}, F_*(\tilde{F}_1^{(p)}/\tilde{F}_1^{(p-1)}))$$

along the special fiber $C \times \{\xi\}$, we obtain a flat family $(\tilde{G'}_1^{(p)}, \tilde{G'}_2^{(p)}, \tilde{\phi'}^{(p)}, \tilde{\nabla'}^{(p)}, F_*(\tilde{G'}_1^{(p)}))$ of $(\boldsymbol{\alpha}'_p, \boldsymbol{\beta}_p, \gamma_p)$-semistable $\Lambda^1_{D(\mathbf{t})}$-triples for some weight $(\boldsymbol{\alpha}'_p, \boldsymbol{\beta}_p, \gamma_p)$. We may assume that $(\boldsymbol{\alpha}'_p, \boldsymbol{\beta}_p, \gamma_p)$-seminstable $\Leftrightarrow$ $(\boldsymbol{\alpha}'_p, \boldsymbol{\beta}_p, \gamma_p)$-stable. This flat family defines a morphism

$$f : \operatorname{Spec} R \longrightarrow \overline{M_C^{D(\mathbf{t}), \boldsymbol{\alpha}'_p, \boldsymbol{\beta}_p, \gamma_p}}(r_p, d_p, \{1\}_{1 \le i \le nr_p}).$$

Note that the moduli space $M_C^{\boldsymbol{\alpha}_p}(\mathbf{t}, \boldsymbol{\lambda}^{(p)})$ of $\boldsymbol{\alpha}_p$-stable $(\mathbf{t}, \boldsymbol{\lambda}^{(p)})$-parabolic connections for a certain weight $\boldsymbol{\alpha}_p$ is a locally closed subscheme of $\overline{M_C^{D(\mathbf{t}), \boldsymbol{\alpha}'_p, \boldsymbol{\beta}_p, \gamma_p}}(r_p, d_p, \{1\}_{1 \le i \le nr_p})$ and the image $f(\operatorname{Spec} K)$ is contained in the moduli space $M_C^{\boldsymbol{\alpha}_p}(\mathbf{t}, \boldsymbol{\lambda}^{(p)})$. Since $(\tilde{G'}_1^{(p)}, \tilde{G'}_2^{(p)}, \tilde{\phi'}^{(p)}, \tilde{\nabla'}^{(p)}) \otimes K$ is equivalent to the irreducible connection $(F^{(p)}/F^{(p-1)}, \overline{\nabla^{(p)}})$ which corresponds to a constant family of the irreducible representation ρ_p, $\mathbf{RH}_{(x, \boldsymbol{\lambda}^{(p)})}(f(\operatorname{Spec} K))$ becomes one point $[\rho_p]$, where $\mathbf{RH}_{(x, \boldsymbol{\lambda}^{(p)})}$ is the morphism

$$\mathbf{RH}_{(x, \boldsymbol{\lambda}^{(p)})} : M_C^{\boldsymbol{\alpha}_p}(\mathbf{t}, \boldsymbol{\lambda}^{(p)}) \longrightarrow RP_{r_p}(C, \mathbf{t})_{\mathbf{a}^{(p)}}$$

determined by Riemann-Hilbert correspondence and $rh(\boldsymbol{\lambda}^{(p)}) = \mathbf{a}^{(p)}$. Since $\mathbf{RH}^{-1}_{(x, \boldsymbol{\lambda}^{(p)})}([\rho_p])$ is compact by the previous proof and $f(\operatorname{Spec} K)$ is contained in $\mathbf{RH}^{-1}_{(x, \boldsymbol{\lambda}^{(p)})}([\rho_p])$, the closure of $f(\operatorname{Spec} K)$ must be contained in $\mathbf{RH}^{-1}_{(x, \boldsymbol{\lambda}^{(p)})}([\rho_p])$. In particular, $f(\xi)$ is contained in $\mathbf{RH}^{-1}_{(x, \boldsymbol{\lambda}^{(p)})}([\rho_p]) \subset M_C^{\boldsymbol{\alpha}_p}(\mathbf{t}, \boldsymbol{\lambda}^{(p)})$. So the special fiber $(\tilde{G'}_1^{(p)}, \tilde{G'}_2^{(p)}, \tilde{\phi'}^{(p)}, \tilde{\nabla'}^{(p)}, F_*(\tilde{G'}_1^{(p)})) \otimes k(\xi)$ becomes a $(\mathbf{t}, \boldsymbol{\lambda}^{(p)})$-parabolic connection which implies that $\tilde{\phi'}^{(p)} \otimes k(\xi)$ must be isomorphic. Then $\overline{\tilde{\phi}^{(p)}} \otimes k(\xi)$ also becomes isomorphic and $\phi' = \tilde{\phi} \otimes k(\xi)$ must be an isomorphism. Let $\{\tilde{l}_j^{(i)}\}$ be the flat family of parabolic structures corresponding to $F_*(\tilde{E}_1)$. Then we have $(\mathsf{res}_{t_i}(\tilde{\nabla}) - \lambda_j^{(i)} \tilde{\phi}|_{t_i \otimes R})(\tilde{l}_j^{(i)}) \subset \tilde{\phi}(\tilde{l}_{j+1}^{(i)})$. Thus $(E'_1, E'_2, \phi', \nabla', F_*(E'_1)) \in M_C^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda})$ which is equivalent to $x_0 \in M_C^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda})$. Therefore $\mathbf{RH}^{-1}_{(x, \boldsymbol{\lambda})}([\rho])$ is stable under specialization. Hence $\mathbf{RH}^{-1}_{(x, \boldsymbol{\lambda})}([\rho])$ becomes a compact subset of $M_C^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda})$. □

Proof of Theorem 2.2. We will prove that

$$\mathbf{RH}_{(x, \boldsymbol{\lambda})} : M_C^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}) \longrightarrow RP_r(C, \mathbf{t})_{\mathbf{a}}$$

is an analytic isomorphism for generic $\boldsymbol{\lambda}$ and gives an analytic resolution of singularities of $RP_r(C, \mathbf{t})_{\mathbf{a}}$ for special $\boldsymbol{\lambda}$. First note that $RP_r(C, \mathbf{t})_{\mathbf{a}}$ is an irreducible variety since by Proposition 5.1 it is the image

of the irreducible variety $M_C^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda})$ by $\mathbf{RH}_{(x,\boldsymbol{\lambda})}$. We set

$$RP_r(C,\mathbf{t})_{\mathbf{a}}^{\mathrm{sing}} := \left\{ [\rho]\in RP_r(C,\mathbf{t})_{\mathbf{a}} \;\middle|\; \begin{array}{l} \rho \text{ is reducible or} \\ \dim\left(\ker\left(\rho(\gamma_i)-\exp(-2\pi\sqrt{-1}\lambda^{(i)}_j)I_r\right)\right)\geq 2 \text{ for some } i,j \end{array} \right\},$$

where γ_i is a loop around t_i for $1\leq i\leq n$. Then we can see that $RP_r(C,\mathbf{t})_{\mathbf{a}}^{\mathrm{sing}}$ is a Zariski closed subset of $RP_r(C,\mathbf{t})_{\mathbf{a}}$. We will see that it is a proper closed subset. Since $\dim M_C^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda})=2r^2(g-1)+nr(r-1)+2$ and $\mathbf{RH}_{(x,\boldsymbol{\lambda})}$ is surjective, we have $\dim RP_r(C,\mathbf{t})_{\mathbf{a}}\leq 2r^2(g-1)+nr(r-1)+2$. On the other hand, $RP_r(C,\mathbf{t})_{\mathbf{a}}$ is a Zariski closed subset of the $r^2(2g+n-2)+1$ dimensional irreducible variety $RP_r(C,\mathbf{t})$ defined by $nr-1$ equations given by

$$\det(XI_r-\rho(\gamma_i)) = X^r + a^{(i)}_{r-1}X^{r-1}+\cdots+a^{(i)}_1X+a^{(i)}_0 \quad (i=1,\ldots,n). \tag{9}$$

Note that the equation $\det(\rho(\gamma_1))\cdots\det(\rho(\gamma_n))=1=(-1)^{nr}a^{(1)}_0\cdots a^{(n)}_0$ is automatically satisfied and so the condition (9) is in fact equivalent to $nr-1$ equations. So we have

$$\dim RP_r(C,\mathbf{t})_{\mathbf{a}}\geq r^2(2g+n-2)+1-(nr-1)=2r^2(g-1)+nr(r-1)+2$$

and then we have $\dim RP_r(C,\mathbf{t})_{\mathbf{a}}=2r^2(g-1)+nr(r-1)+2$. First consider the locus $RP_r(C,\mathbf{t})_{\mathbf{a}}^{\mathrm{irr}}\cap RP_r(C,\mathbf{t})_{\mathbf{a}}^{\mathrm{sing}}$ in $RP_r(C,\mathbf{t})_{\mathbf{a}}^{\mathrm{sing}}$ corresponding to the irreducible representations. By Proposition 3.1, we can obtain an isomorphism

$$\sigma:\mathcal{M}_C(\mathbf{t},\boldsymbol{\lambda})\xrightarrow{\sim}\mathcal{M}_C(\mathbf{t},\boldsymbol{\mu}),$$

where $0\leq \mathrm{Re}(\mu^{(i)}_j)<1$ for any i,j and $\mathcal{M}_C(\mathbf{t},\boldsymbol{\lambda})$ is the moduli stack of $(\mathbf{t},\boldsymbol{\lambda})$-parabolic connections and so on. σ also induces an isomorphism

$$\sigma: M_C^{\mathrm{irr}}(\mathbf{t},\boldsymbol{\lambda})\xrightarrow{\sim}M_C^{\mathrm{irr}}(\mathbf{t},\boldsymbol{\mu}).$$

Take any point $[\rho]\in RP_r(C,\mathbf{t})_{\mathbf{a}}^{\mathrm{irr}}\cap RP_r(C,\mathbf{t})_{\mathbf{a}}^{\mathrm{sing}}$. By [[4], Proposition 5.4], we can take a unique pair (E,∇) of a vector bundle E on C and a logarithmic connection $\nabla:E\to E\otimes\Omega^1_C(t_1+\cdots+t_n)$ such that $\ker\nabla^{an}|_{C\setminus\{t_1,\ldots,t_n\}}$ corresponds to the representation ρ and all the eigenvalues of $\mathsf{res}_{t_i}(\nabla)$ lies in $\{z\in\mathbf{C}|0\leq\mathrm{Re}(z)<1\}$. Then the fiber $\mathbf{RH}^{-1}_{(x,\boldsymbol{\mu})}([\rho])$ is isomorphic to the moduli space of parabolic structures $\{l^{(i)}_j\}$ on E whose dimension should be positive because $\mu^{(i)}_j=\mu^{(i)}_{j'}$ for some i and $j\neq j'$ and the automorphisms of (E,∇) are only scalar multiplications. So the fiber $\mathbf{RH}^{-1}_{(x,\boldsymbol{\lambda})}([\rho])$ also has positive dimension. For a generic point ξ of $RP_r(C,\mathbf{t})_{\mathbf{a}}^{\mathrm{irr}}\cap RP_r(C,\mathbf{t})_{\mathbf{a}}^{\mathrm{sing}}$, we have

$$\begin{aligned}\dim\mathbf{RH}^{-1}_{(x,\boldsymbol{\lambda})}(\xi)+\dim_{\xi}\left(RP_r(C,\mathbf{t})_{\mathbf{a}}^{\mathrm{irr}}\cap RP_r(C,\mathbf{t})_{\mathbf{a}}^{\mathrm{sing}}\right)&\leq\dim M_C^{\mathrm{irr}}(\mathbf{t},\boldsymbol{\lambda})\\&=2r^2(g-1)+nr(r-1)+2.\end{aligned}$$

Thus we have $\dim\left(RP_r(C,\mathbf{t})_{\mathbf{a}}^{\mathrm{irr}}\cap RP_r(C,\mathbf{t})_{\mathbf{a}}^{\mathrm{sing}}\right)<2r^2(g-1)+nr(r-1)+2=\dim RP_r(C,\mathbf{t})_{\mathbf{a}}$. Next we consider the locus $RP_r(C,\mathbf{t})_{\mathbf{a}}^{\mathrm{red}}\subset RP_r(C,\mathbf{t})_{\mathbf{a}}^{\mathrm{sing}}$ corresponding to the reducible representations. For a point $[\rho]$ of $RP_r(C,\mathbf{t})_{\mathbf{a}}^{\mathrm{red}}$ we may write $\rho=\rho_1\oplus\rho_2$. If ρ_i is a representation of dimension r_i and its local exponent data is $\mathbf{a}_i$ for $i=1,2$, then the set of such $[\rho]$ can be parameterized by $RP_{r_1}(C,\mathbf{t})_{\mathbf{a}_1}\times RP_{r_2}(C,\mathbf{t})_{\mathbf{a}_2}$. Note that the assumption $rn-2(r+1)>0$ for $g=0$, $n\geq 2$ for $g=1$ and $n\geq 1$ for $g\geq 2$ implies

$$\begin{aligned}\dim\left(RP_{r_1}(C,\mathbf{t})_{\mathbf{a}_1}\times RP_{r_2}(C,\mathbf{t})_{\mathbf{a}_2}\right)&=2r_1^2(g-1)+nr_1(r_1-1)+2+2r_2^2(g-1)+nr_2(r_2-1)+2\\&\leq 2(r_1+r_2)^2(g-1)+n(r_1+r_2)(r_1+r_2-1)\\&=\dim RP_r(C,\mathbf{t})_{\mathbf{a}}-2.\end{aligned}$$

Since $RP_r(C,\mathbf{t})_{\mathbf{a}}^{\mathrm{red}}$ can be covered by a finite numbers of the images of such spaces, we have

$$\dim RP_r(C,\mathbf{t})_{\mathbf{a}}^{\mathrm{red}}<2r^2(g-1)+nr(r-1)+2=\dim RP_r(C,\mathbf{t})_{\mathbf{a}}.$$

Thus $RP_r(C,\mathbf{t})_{\mathbf{a}}^{\mathrm{sing}}$ becomes a proper closed subset of $RP_r(C,\mathbf{t})_{\mathbf{a}}$. If we set $RP_r(C,\mathbf{t})_{\mathbf{a}}^{\sharp}:=RP_r(C,\mathbf{t})_{\mathbf{a}}\setminus RP_r(C,\mathbf{t})_{\mathbf{a}}^{\mathrm{sing}}$, then it becomes a non-empty Zariski open subset of $RP_r(C,\mathbf{t})_{\mathbf{a}}$. We put $M_C^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda})^{\sharp}:=\mathbf{RH}^{-1}_{(x,\boldsymbol{\lambda})}(RP_r(C,\mathbf{t})_{\mathbf{a}}^{\sharp})$ and consider the restriction

$$\mathbf{RH}_{(x,\boldsymbol{\lambda})}|_{M_C^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda})^{\sharp}}:M_C^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda})^{\sharp}\longrightarrow RP_r(C,\mathbf{t})_{\mathbf{a}}^{\sharp}.$$

Recall that σ induces an isomorphism $\sigma : M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda})^\sharp \xrightarrow{\sim} M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\mu})^\sharp$ which is compatible with $\mathbf{RH}$. For any point $[\rho]$ of $RP_r(C,\mathbf{t})^\sharp_{\mathbf{a}}$, we can see by [[4], Proposition 5.4] that there is a unique $(\mathbf{t},\boldsymbol{\mu})$-parabolic connection $(E,\nabla,\{l^{(i)}_j\})$ such that $\mathbf{RH}_{(x,\boldsymbol{\mu})}((E,\nabla,\{l^{(i)}_j\})=[\rho]$ because the parabolic structure $\{l^{(i)}_j\}$ is uniquely determined by (E,∇). So $\mathbf{RH}_{(x,\boldsymbol{\mu})}$ gives a one to one correspondence between the points of $M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\mu})^\sharp$ and the points of $RP_r(C,\mathbf{t})^\sharp_{\mathbf{a}}$. We can extend this correspondence to a correspondence between flat families. Thus the morphism $M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\mu})^\sharp \xrightarrow{\mathbf{RH}_{(x,\boldsymbol{\mu})}} RP_r(C,\mathbf{t})^\sharp_{\mathbf{a}}$ becomes an isomorphism and so $M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda})^\sharp \xrightarrow{\mathbf{RH}_{(x,\boldsymbol{\lambda})}} RP_r(C,\mathbf{t})^\sharp_{\mathbf{a}}$ is also an isomorphism. Hence the morphism $\mathbf{RH}_{(x,\boldsymbol{\lambda})} : M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda}) \longrightarrow RP_r(C,\mathbf{t})_{\mathbf{a}}$ becomes a bimeromorphic morphism. If $\boldsymbol{\lambda}$ is generic, we have $M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda})^\sharp = M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda})$, $RP_r(C,\mathbf{t})^\sharp_{\mathbf{a}} = RP_r(C,\mathbf{t})_{\mathbf{a}}$, and so

$$\mathbf{RH}_{(x,\boldsymbol{\lambda})} : M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda}) \longrightarrow RP_r(C,\mathbf{t})_{\mathbf{a}}$$

is an analytic isomorphism.

In order to prove the properness of $\mathbf{RH}_{(x,\boldsymbol{\lambda})}$, we will use the following lemma due to A. Fujiki:

Lemma 5.2. **([7], Lemma 10.3)** *Let $f : X \to Y$ be a surjective holomorphic mapping of irreducible analytic varieties. Assume that an analytic closed subset S of Y exists such that* $\operatorname{codim}_Y S \geq 2$*, $X^\sharp := f^{-1}(Y^\sharp)$ is dense in X, where $Y^\sharp = Y \setminus S$ and that the restriction $f|_{X^\sharp} : X^\sharp \to Y^\sharp$ is an analytic isomorphism. Moreover assume that the fibers $f^{-1}(y)$ are compact for all $y \in Y$. Then f is a proper mapping.*

Applying the above lemma and using Proposition 5.1 and Proposition 5.2, it suffices to prove that $\operatorname{codim}_{RP_r(C,\mathbf{t})_{\mathbf{a}}}(RP_r(C,\mathbf{t})^{\mathrm{sing}}_{\mathbf{a}}) \geq 2$ in order to obtain the properness of $\mathbf{RH}_{(x,\boldsymbol{\lambda})}$. Recall that we have $\dim RP_r(C,\mathbf{t})^{\mathrm{red}}_{\mathbf{a}} \leq \dim RP_r(C,\mathbf{t})_{\mathbf{a}} - 2$. On the other hand, $M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda})^\sharp$ is a non-empty Zariski open subset of $M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda})$ and so we have $\dim\left(M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda}) \setminus M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda})^\sharp\right) \leq \dim M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda}) - 1$. Since the dimension of every fiber of the surjective morphism

$$M^{\mathrm{irr}}_C(\mathbf{t},\boldsymbol{\lambda}) \setminus M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda})^\sharp \xrightarrow{\mathbf{RH}_{(x,\boldsymbol{\lambda})}} RP_r(C,\mathbf{t})^{\mathrm{sing}}_{\mathbf{a}} \cap RP_r(C,\mathbf{t})^{\mathrm{irr}}_{\mathbf{a}}$$

has positive dimension, we have

$$\begin{aligned}\dim\left(RP_r(C,\mathbf{t})^{\mathrm{sing}}_{\mathbf{a}} \cap RP_r(C,\mathbf{t})^{\mathrm{irr}}_{\mathbf{a}}\right) &\leq \dim\left(M^{\mathrm{irr}}_C(\mathbf{t},\boldsymbol{\lambda}) \setminus M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda})^\sharp\right) - 1 \\ &\leq \dim M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda}) - 1 - 1 = \dim RP_r(C,\mathbf{t})_{\mathbf{a}} - 2.\end{aligned}$$

Thus we have $\operatorname{codim}_{RP_r(C,\mathbf{t})_{\mathbf{a}}}(RP_r(C,\mathbf{t})^{\mathrm{sing}}_{\mathbf{a}}) \geq 2$ and obtain the properness of $\mathbf{RH}_{(x,\boldsymbol{\lambda})}$ by Lemma 5.2. In particular, $\mathbf{RH}_{(x,\boldsymbol{\lambda})} : M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda}) \to RP_r(C,\mathbf{t})_{\mathbf{a}}$ gives an analytic resolution of singularities of $RP_r(C,\mathbf{t})_{\mathbf{a}}$ for special $\boldsymbol{\lambda}$. By the same argument, we can see that the morphism

$$\mathbf{RH} : M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d) \longrightarrow RP_r(\mathcal{C},\tilde{\mathbf{t}}) \times_{\mathcal{A}^{(n)}_r} \Lambda^{(n)}_r(d)$$

is also a proper mapping. □

Remark 5.2. Under the assumption of Theorem 2.2, $RP_r(C,\mathbf{t})_{\mathbf{a}}$ is a normal variety and the locus

$$RP_r(C,\mathbf{t})^{\mathrm{sing}}_{\mathbf{a}} = \left\{ [\rho] \in RP_r(C,\mathbf{t})_{\mathbf{a}} \;\middle|\; \begin{array}{l} \rho \text{ is reducible or} \\ \dim\left(\ker\left(\rho(\gamma_i) - \exp(-2\pi\sqrt{-1}\lambda^{(i)}_j)I_r\right)\right) \geq 2 \text{ for some } i,j \end{array} \right\}$$

is just the singular locus of $RP_r(C,\mathbf{t})_{\mathbf{a}}$.

Proof. Note that $RP_r(C,\mathbf{t})$ is a Cohen-Macaulay irreducible variety by [[5], Theorem 4 (2)]. In the proof of Theorem 2.2, we showed that $RP_r(C,\mathbf{t})_{\mathbf{a}}$ a Zariski closed subset of $RP_r(C,\mathbf{t})$ defined by $nr-1$ equations and it is of codimension $nr-1$ in $RP_r(C,\mathbf{t})$. Thus $RP_r(C,\mathbf{t})_{\mathbf{a}}$ is Cohen-Macaulay. Since $RP_r(C,\mathbf{t})^\sharp_{\mathbf{a}}$ is isomorphic to $M^{\boldsymbol{\alpha}}_C(C,\mathbf{t})^\sharp$, $RP_r(C,\mathbf{t})^\sharp_{\mathbf{a}}$ is smooth. So $RP_r(C,\mathbf{t})_{\mathbf{a}}$ is regular in codimension one because $\operatorname{codim}_{RP_r(C,\mathbf{t})_{\mathbf{a}}}(RP_r(C,\mathbf{t})^{\mathrm{sing}}_{\mathbf{a}}) \geq 2$. Therefore $RP_r(C,\mathbf{t})_{\mathbf{a}}$ is a normal variety by Serre's criterion for normality.

We will prove that every fiber of $\mathbf{RH}_{(x,\boldsymbol{\lambda})}$ over a point of $RP_r(C,\mathbf{t})^{\mathrm{sing}}_{\mathbf{a}}$ has positive dimension. For a point $[\rho] \in RP_r(C,\mathbf{t})^{\mathrm{irr}}_{\mathbf{a}} \cap RP_r(C,\mathbf{t})^{\mathrm{sing}}_{\mathbf{a}}$, we have already showed in the proof of Theorem 2.2 that $\dim \mathbf{RH}^{-1}_{(x,\boldsymbol{\lambda})}([\rho]) \geq 1$. So take a point $[\rho] \in RP_r(C,\mathbf{t})^{\mathrm{red}}_{\mathbf{a}}$. By Proposition 3.1, we can take an isomorphism

$$\sigma : \mathcal{M}_C(\mathbf{t},\boldsymbol{\lambda}) \xrightarrow{\sim} \mathcal{M}_C(\mathbf{t},\boldsymbol{\mu})$$

of moduli stacks by composing elementary transformations and so on, where $0 \leq \mathrm{Re}(\mu^{(i)}_j) < 1$ for any i, j.

First we assume that ρ is an extention of two irreducible representations ρ_1, ρ_2 and that $\mu^{(i)}_j \neq \mu^{(i)}_{j'}$ for any $j \neq j'$ and any i. We denote the dimension of ρ_i by r_i. Since $\mathbf{RH}_{(x,\boldsymbol{\lambda})}$ is surjective by Proposition 5.1, there is an $\boldsymbol{\alpha}$-stable $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connection $(E, \nabla_E, \{l^{(i)}_j\})$ such that $\mathbf{RH}_{(x,\boldsymbol{\lambda})}(E, \nabla_E, \{l^{(i)}_j\}) = [\rho]$. We put $\sigma(E, \nabla_E, \{l^{(i)}_j\}) = (E', \nabla_{E'}, \{l'^{(i)}_j\})$. Since $\mu^{(i)}_j \neq \mu^{(i)}_{j'}$ for any $j \neq j'$ and any i, the parabolic structure $\{l'^{(i)}_j\}$ is uniquely determined by $(E', \nabla_{E'})$. So we have

$$\mathrm{End}(E', \nabla_{E'}) = \mathrm{End}(E', \nabla_{E'}, \{l'^{(i)}_j\}) \cong \mathrm{End}(E, \nabla_E, \{l^{(i)}_j\})$$

which is isomorphic to $\mathbf{C}$ because $\mathrm{End}(E, \nabla_E, \{l^{(i)}_j\})$ is $\boldsymbol{\alpha}$-stable. Then $(E', \nabla_{E'})$ can not split. Thus the representation $\rho_{E'}$ corresponding to $\ker \nabla^{an}_{E'}|_{C\setminus\{t_1,\ldots,t_n\}}$ can not split and so we may assume that $\rho_{E'}$ is given by matrices

$$\rho_{E'}(\alpha_k) = \begin{pmatrix} \rho_1(\alpha_k) & A^{(0)}_k \\ 0 & \rho_2(\alpha_k) \end{pmatrix} \quad \rho_{E'}(\beta_k) = \begin{pmatrix} \rho_1(\beta_k) & B^{(0)}_k \\ 0 & \rho_2(\beta_k) \end{pmatrix}$$
$$\rho_{E'}(\gamma_i) = \begin{pmatrix} \rho_1(\gamma_i) & C^{(0)}_i \\ 0 & \rho_2(\gamma_i) \end{pmatrix} \quad (1 \leq k \leq g,\ 1 \leq i \leq n-1).$$

Note that $\rho_{E'}(\gamma_n)$ is uniquely determined by the above data. Consider the family of representations over $M(r_1, r_2, \mathbf{C})^{2g+n-1}$ given by matrices

$$\tilde{\rho}(\alpha_k) = \begin{pmatrix} \rho_1(\alpha_k) & A^{(0)}_k + A_k \\ 0 & \rho_2(\alpha_k) \end{pmatrix} \quad \tilde{\rho}(\beta_k) = \begin{pmatrix} \rho_1(\beta_k) & B^{(0)}_k + B_k \\ 0 & \rho_2(\beta_k) \end{pmatrix}$$
$$\tilde{\rho}(\gamma_i) = \begin{pmatrix} \rho_1(\gamma_i) & C^{(0)}_i + C_i \\ 0 & \rho_2(\gamma_i) \end{pmatrix} \quad (1 \leq k \leq g,\ 1 \leq i \leq n-1),$$

where matrices $A_k, B_k (1 \leq k \leq g), C_i (1 \leq i \leq n-1)$ move around in $M(r_1, r_2, \mathbf{C})^{2g+n-1}$. Applying the relative version of [[4], Proposition 5.4] and the transform σ^{-1}, we can obtain an analytic flat family $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}^{(i)}_j\})$ of $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connections on $C \times M(r_1, r_2, \mathbf{C})^{2g+n-1}$ over $M(r_1, r_2, \mathbf{C})^{2g+n-1}$ which corresponds to $\tilde{\rho}$. We denote the point of $M(r_1, r_2, \mathbf{C})^{2g+n-1}$ corresponding to $A_k = B_k = C_i = 0$ $(1 \leq k \leq g, 1 \leq i \leq n-1)$ by p_0. Then the fiber of $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}^{(i)}_j\})$ over p_0 is just $(E, \nabla_E, \{l^{(i)}_j\})$ which is $\boldsymbol{\alpha}$-stable. So there is an analytic open neighborhood U of p_0 such that every fiber of $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}^{(i)}_j\})$ over a point of U is $\boldsymbol{\alpha}$-stable. Then the family $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}^{(i)}_j\})_U$ induces a morphism $f : U \to M^{\boldsymbol{\alpha}}_C(\mathbf{t}, \boldsymbol{\lambda})$ such that $f(U) \subset \mathbf{RH}^{-1}_{(x,\boldsymbol{\lambda})}([\rho])$. The group

$$G = \left\{ \begin{pmatrix} uI_{r_1} & V \\ 0 & I_{r_2} \end{pmatrix} \middle| \begin{array}{l} u \in \mathbf{C}^{\times} \\ V \in M(r_1, r_2, \mathbf{C}) \end{array} \right\}$$

acts on $M(r_1, r_2, \mathbf{C})^{2g+n-1}$ by the adjont action. If $f(x) = f(y)$, then there is an element $g \in G$ such that $x = gy$. So we have $\dim f(U) \geq (2g + n - 1)r_1 r_2 - (r_1 r_2 + 1) \geq 1$, since we assume $rn - 2r - 2 > 0$ if $g = 0$, $n \geq 2$ if $g = 1$ and $n \geq 1$ if $g \geq 2$. In particular, we have $\dim \mathbf{RH}^{-1}_{(x,\boldsymbol{\lambda})}([\rho]) \geq 1$.

Next we consider the case $r \geq 3$ or $n \geq 2$ and no assumption on the reducible representation ρ. We may assume that ρ is an extension of two representations ρ_1, ρ_2 and denote the dimension of ρ_i by r_i. For each $p = 1, 2$, there is an $\boldsymbol{\alpha}_p$-stable $(\mathbf{t}, \boldsymbol{\mu}_p)$-parabolic connection $(E_p, \nabla_{E_p}, \{(l_p)^{(i)}_j\})$ for some weight $\boldsymbol{\alpha}_p$ such that $\mathbf{RH}_{(x,\boldsymbol{\mu}_p)}(E_p, \nabla_{E_p}, \{(l_p)^{(i)}_j\}) = [\rho_p]$ and $0 \leq \mathrm{Re}((\mu_p)^{(i)}_j) < 1$ for any i, j. We put $d_p := \deg E_p$. If we put $\Delta := \{z \in \mathbf{C} | |z| < \epsilon\}$ $(\epsilon > 0)$, then we can take morphisms

$$m_p : \Delta \longrightarrow \Lambda^{(n)}_{r_p}(d_p) \quad (p = 1, 2)$$

such that $m_p(0) = \boldsymbol{\mu}_p$ and for $m_p(t) = \{(\mu'_p)^{(i)}_j\}$ $(t \in \Delta \setminus \{0\})$, we have $(\mu'_p)^{(i)}_j \neq (\mu'_{p'})^{(i)}_{j'}$ for $(p, i, j) \neq (p', i, j')$. Replacing Δ by a neighborhood of 0, we can take a morphism $\varphi_p : \Delta \longrightarrow M^{\boldsymbol{\alpha}_p}_C(\mathbf{t}, r_p, d_p)$ which is a lift of m_p and satisfies $\varphi_p(0) = (E_p, \nabla_{E_p}, \{(l_p)^{(i)}_j\})$. The composite $\mathbf{RH} \circ \varphi_p$ determines a family $\tilde{\rho}_p$ of representations over Δ and $\tilde{\rho}_1 \oplus \tilde{\rho}_2$ determines a morphism $\varphi : \Delta \to RP_r(C, \mathbf{t})$ such that $\varphi(0) = [\rho]$ and for $t \in \Delta \setminus \{0\}$, the representation corresponding to $\varphi(t)$ satisfies the condition considered in the former case. Then we have $\dim \mathbf{RH}^{-1}(\varphi(t)) \geq 1$ for any $t \in \Delta \setminus \{0\}$ by the proof of the former case. Since the base

change $\mathbf{RH}_\Delta : M^{\boldsymbol{\alpha}}_C(\mathbf{t}, r, d)_\Delta \to \Delta$ is a proper morphism, we can see by the upper semi-continuity of the fiber dimension of a holomorphic mapping that $\dim(\mathbf{RH}_\Delta)^{-1}(0) \geq 1$, which means $\dim \mathbf{RH}^{-1}_{(x,\boldsymbol{\lambda})}([\rho]) \geq 1$.

Finally assume that $r = 2$ and $n = 1$. In this case we have $g \geq 2$ by assumption of Theorem 2.2. By Proposition 5.1, we can take an $\boldsymbol{\alpha}$-stable $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connection $(E, \nabla, \{l^{(i)}_j\})$ such that $\mathbf{RH}(E, \nabla, \{l^{(i)}_j\}) = [\rho]$. We put $\sigma(E, \nabla, \{l^{(i)}_j\}) = (E', \nabla', \{(l')^{(i)}_j\})$. Note that the local monodromy $\rho_i(\gamma_1)$ is trivial for $i = 1, 2$, because ρ_i is a one dimensional representation and $n = 1$. So the representation $\rho_{E'}$ corresponding to $\ker(\nabla')^{an}|_{C\setminus\{t_1,\dots,t_n\}}$ can be given by the following data:

$$\rho_{E'}(\alpha_k) = \begin{pmatrix} \rho_1(\alpha_k) & a^{(0)}_k \\ 0 & \rho_2(\alpha_k) \end{pmatrix} \quad \rho_{E'}(\beta_k) = \begin{pmatrix} \rho_1(\beta_k) & b^{(0)}_k \\ 0 & \rho_2(\beta_k) \end{pmatrix} \quad (1 \leq k \leq g).$$

Note that the monodromy matrix

$$\rho_{E'}(\gamma_1) = \begin{pmatrix} 1 & c^{(0)} \\ 0 & 1 \end{pmatrix}$$

is given by

$$\rho_g(\beta_g)^{-1}\rho_g(\alpha_g)^{-1}\rho_g(\beta_g)\rho_g(\alpha_g)\cdots\rho_1(\beta_g)^{-1}\rho_1(\alpha_g)^{-1}\rho_g(\beta_g)\rho_g(\alpha_g).$$

By the openness of stability, we may assume that $\rho_1 \not\cong \rho_2$. If $c^{(0)} \neq 0$, then we can prove in the same manner as the first case that $\dim_{(E,\nabla,\{l^{(i)}_j\})} \mathbf{RH}^{-1}_{(x,\boldsymbol{\lambda})}([\rho]) \geq 2g - 2 > 0$. If $c^{(0)} = 0$, then we may assume by the openness of stability that $l^{(1)}_1 \neq \left\{\binom{*}{0}\right\}$. Consider the family of representations given by matrices

$$\tilde{\rho}(\alpha_k) = \begin{pmatrix} \rho_1(\alpha_k) & a^{(0)}_k + a_k \\ 0 & \rho_2(\alpha_k) \end{pmatrix} \quad \tilde{\rho}(\beta_k) = \begin{pmatrix} \rho_1(\beta_k) & b^{(0)}_k + b_k \\ 0 & \rho_2(\beta_k) \end{pmatrix} \quad (1 \leq k \leq g),$$

where a_k, b_k $(1 \leq k \leq g)$ move around in $\{c_k = 0\} \subset \mathbf{C}^{2g}$. Here c_k is given by

$$\begin{pmatrix} 1 & c_k \\ 0 & 1 \end{pmatrix} = \tilde{\rho}(\beta_g)^{-1}\tilde{\rho}(\alpha_g)^{-1}\tilde{\rho}(\beta_g)\tilde{\rho}(\alpha_g)\cdots\tilde{\rho}(\beta_1)^{-1}\tilde{\rho}(\alpha_1)^{-1}\tilde{\rho}(\beta_1)\tilde{\rho}(\alpha_1).$$

Adding parabolic structure, we can obtain a family $(\tilde{E}', \tilde{\nabla}', \{(\tilde{l}')^{(i)}_j\})$ of $(\mathbf{t}, \boldsymbol{\mu})$-parabolic connections over a variety Y of dimension at least $2g$. If we put

$$Y' := \left\{ y \in Y \,\middle|\, \sigma^{-1}(\tilde{E}', \tilde{\nabla}', \{(\tilde{l}')^{(i)}_j\}) \otimes k(y) \text{ is } \boldsymbol{\alpha}\text{-stable} \right\}$$

we obtain a morphism $f' : Y' \to M^{\boldsymbol{\alpha}}_C(\mathbf{t}, \boldsymbol{\lambda})$. We can see that $f'(y_1) = f'(y_2)$ if there is an isomorphism $\varphi : (\tilde{E}', \tilde{\nabla}') \otimes k(y_1) \xrightarrow{\sim} (\tilde{E}', \tilde{\nabla}') \otimes k(y_2)$ such that $\varphi((\tilde{l}')^{(i)}_j \otimes k(y_1)) = (\tilde{l}')^{(i)}_j \otimes k(y_2)$. The isomorphism φ corresponds to an isomorphisms between the representations $\tilde{\rho}_{y_1}, \tilde{\rho}_{y_2}$ which is given by an element of the group

$$\left\{ \begin{pmatrix} c & a \\ 0 & 1 \end{pmatrix} \,\middle|\, c \in \mathbf{C}^\times,\ a \in \mathbf{C} \right\}.$$

So we have $\dim f'(Y') = \dim Y' - 2 \geq 2g - 2 > 0$. Since $f'(Y') \subset \mathbf{RH}^{-1}_{(x,\boldsymbol{\lambda})}([\rho])$, we have $\mathbf{RH}^{-1}_{(x,\boldsymbol{\lambda})}([\rho]) > 0$.

From all the above argument, we have in any case that $\dim \mathbf{RH}^{-1}([\rho]) \geq 1$ for any $[\rho] \in RP_r(C, \mathbf{t})^{\mathrm{sing}}_{\mathbf{a}}$. Now take any point $p \in RP_r(C, \mathbf{t})^{\mathrm{sing}}_{\mathbf{a}}$. We want to show that p is a singular point of $RP_r(C, \mathbf{t})_{\mathbf{a}}$. Assume that p is a non-singular point of $RP_r(C, \mathbf{t})_{\mathbf{a}}$. By Proposition 6.2, there is a non-degenerate 2-form ω on $M^{\boldsymbol{\alpha}}_C(\mathbf{t}, \boldsymbol{\lambda})$. Via an isomorphism $M^{\boldsymbol{\alpha}}_C(\mathbf{t}, \boldsymbol{\lambda})^\sharp \xrightarrow{\mathbf{RH}} RP_r(C, \mathbf{t})^\sharp_{\mathbf{a}}$, we can obtain a non-degenerate 2-form ω' on $RP_r(C, \mathbf{t})^\sharp_{\mathbf{a}}$ such that $\mathbf{RH}^*_{(x,\boldsymbol{\lambda})}(\omega') = \omega|_{M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda})^\sharp}$. ω' can be extended to a form defined also in a neighborhood of p because $\mathrm{codim}_{RP_r(C,\mathbf{t})_{\mathbf{a}}}(RP_r(C, \mathbf{t})^{\mathrm{sing}}_{\mathbf{a}}) \geq 2$ and we are assuming that p is a non-singular point. Since $\dim_x \mathbf{RH}^{-1}_{(x,\boldsymbol{\lambda})}(p) \geq 1$ for a point $x \in \mathbf{RH}^{-1}_{(x,\boldsymbol{\lambda})}(p)$, there is a tangent vector $0 \neq v \in \Theta_{M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda}),x}$ such that $(\mathbf{RH}_{(x,\boldsymbol{\lambda})})_*(v) = 0$. For any tangent vector $w \in \Theta_{M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda}),x}$, we have

$$\omega(v, w) = (\mathbf{RH}^*_{(x,\boldsymbol{\lambda})}\,\omega')(v, w) = \omega'((\mathbf{RH}_{(x,\boldsymbol{\lambda})})_*(v), (\mathbf{RH}_{(x,\boldsymbol{\lambda})})_*(w)) = 0.$$

Since ω determines a non-degenerate pairing on the tangent space $\Theta_{M^{\boldsymbol{\alpha}}_C(\mathbf{t},\boldsymbol{\lambda}),x}$, we have $v = 0$ which is a contradiction. Thus p is a singular point of $RP_r(C, \mathbf{t})_{\mathbf{a}}$. $\square$

6. Symplectic structure on the moduli space

Proposition 6.1. *Put $(C,\mathbf{t}) = (\mathcal{C}_x, \tilde{\mathbf{t}}_x)$ for a point $x \in T$ and take a point $\boldsymbol{\lambda} \in \Lambda_r^{(n)}(d)$. Then the moduli space $M_C^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda})$ has a holomorphic symplectic structure (more precisely an algebraic symplectic structure).*

We can obtain the above Proposition by the following two propositions.

Proposition 6.2. *There is a nondegenerate relative 2-form $\omega \in H^0(M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d), \Omega^2_{M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)/T\times\Lambda_r^{(n)}(d)})$.*

Proof. Take a universal family $(\tilde{E},\tilde{\nabla},\{\tilde{l}^{(i)}_j\})$ on $\mathcal{C}\times_T M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)$. We define a complex $\mathcal{F}^\bullet$ by

$$\mathcal{F}^0 = \left\{ s \in \mathcal{E}nd(\tilde{E}) \,\middle|\, s|_{\tilde{t}_i\times M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)}(\tilde{l}^{(i)}_j) \subset \tilde{l}^{(i)}_j \text{ for any } i,j \right\}$$
$$\mathcal{F}^1 = \left\{ s \in \mathcal{E}nd(\tilde{E})\otimes\Omega^1_{\mathcal{C}/T}(D(\tilde{\mathbf{t}})) \,\middle|\, \mathrm{res}_{\tilde{t}_i\times M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)}(s)(\tilde{l}^{(i)}_j) \subset \tilde{l}^{(i)}_{j+1} \text{ for any } i,j \right\} \quad \text{and}$$
$$\nabla_{\mathcal{F}^\bullet} : \mathcal{F}^0 \ni s \mapsto \tilde{\nabla}\circ s - s\circ\tilde{\nabla} \in \mathcal{F}^1.$$

Notice that there is an isomorphism of sheaves

$$\mathbf{R}^1(\pi_{M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)})_*(\mathcal{F}^\bullet) \xrightarrow{\sim} \Theta_{M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)/T\times\Lambda_r^{(n)}(d)},$$

where $\pi_{M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)} : \mathcal{C}\times_T M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d) \to M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)$ is the projection and $\Theta_{M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)/T\times\Lambda_r^{(n)}(d)}$ is the relative algebraic tangent bundle on the moduli space. For each affine open subset $U \subset M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)$, we define a pairing

$$\mathbf{H}^1(\mathcal{C}\times_T U, \mathcal{F}^\bullet_U)\otimes\mathbf{H}^1(\mathcal{C}\times_T U,\mathcal{F}^\bullet_U) \longrightarrow \mathbf{H}^2(\mathcal{C}\times_T U, \Omega^\bullet_{\mathcal{C}\times_T U/U}) \cong H^0(\mathcal{O}_U)$$
$$[\{u_{\alpha\beta}\},\{v_\alpha\}]\otimes[\{u'_{\alpha\beta}\},\{v'_\alpha\}] \mapsto [\{\mathrm{Tr}(u_{\alpha\beta}\circ v'_\beta) - \mathrm{Tr}(v_\alpha\circ u'_{\alpha\beta})\} - \{\mathrm{Tr}(u_{\alpha\beta}\circ u'_{\beta\gamma})\}],$$

where we consider in Čech cohomology with respect to an affine open covering $\{U_\alpha\}$ of $\mathcal{C}\times_T U$, $\{u_{\alpha\beta}\} \in C^1(\mathcal{F}^0)$, $\{v_\alpha\}\in C^0(\mathcal{F}^1)$ and so on. This pairing determines a pairing

$$\omega : \mathbf{R}^1(\pi_{M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)})_*(\mathcal{F}^\bullet)\otimes\mathbf{R}^1(\pi_{M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)})_*(\mathcal{F}^\bullet) \longrightarrow \mathcal{O}_{M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)}. \tag{10}$$

Take any point $x \in M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)$ which corresponds to a $(\mathbf{t},\boldsymbol{\lambda})$-parabolic connection $(E,\nabla,\{l^{(i)}_j\})$. A tangent vector $v \in \Theta_{M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)/T\times\Lambda_r^{(n)}(d)}|_x$ corresponds to a flat family of $(\mathbf{t},\boldsymbol{\lambda})$-parabolic connections $(\tilde{E},\tilde{\nabla},\{\tilde{l}^{(i)}_j\})$ on $\mathcal{C}_x\times\operatorname{Spec}\mathbf{C}[\epsilon]/(\epsilon^2)$ over $\mathbf{C}[\epsilon]/(\epsilon^2)$ which is a lift of $(E,\nabla,\{l^{(i)}_j\})$. We can see that $\omega|_x(v\otimes v) \in \mathbf{H}^0(\Omega^\bullet_{\mathcal{C}_x}) \xrightarrow[\sim]{\mathrm{Tr}^{-1}} \mathbf{H}^2(\mathcal{F}^\bullet_x)$ is just the obstruction class for the lifting of $(\tilde{E},\tilde{\nabla},\{\tilde{l}^{(i)}_j\})$ to a flat family over $\mathbf{C}[\epsilon]/(\epsilon^3)$. Since the moduli space $M^{\boldsymbol{\alpha}}_{\mathcal{C}_x}(\mathbf{t},\boldsymbol{\lambda})$ is smooth, this obstruction class vanishes: $\omega|_x(v\otimes v) = 0$. Thus ω is skew symmetric and determines a 2-form. For each point $x\in M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)$, let

$$\mathbf{H}^1(\mathcal{F}^\bullet_x) \xrightarrow{\xi} \mathbf{H}^1(\mathcal{F}^\bullet_x)^\vee$$

be the homomorphism induced by ω. Then we have the following exact commutative diagram

$$\begin{array}{ccccccccc}
H^0(\mathcal{F}^0_x) & \longrightarrow & H^0(\mathcal{F}^1_x) & \longrightarrow & \mathbf{H}^1(\mathcal{F}^\bullet_x) & \longrightarrow & H^1(\mathcal{F}^0_x) & \longrightarrow & H^1(\mathcal{F}^1_x) \\
{\scriptstyle b_1}\downarrow & & {\scriptstyle b_2}\downarrow & & {\scriptstyle \xi}\downarrow & & {\scriptstyle b_3}\downarrow & & {\scriptstyle b_4}\downarrow \\
H^1(\mathcal{F}^1_x)^\vee & \longrightarrow & H^1(\mathcal{F}^0_x)^\vee & \longrightarrow & \mathbf{H}^1(\mathcal{F}^\bullet_x)^\vee & \longrightarrow & H^0(\mathcal{F}^1_x)^\vee & \longrightarrow & H^0(\mathcal{F}^0_x)^\vee,
\end{array}$$

where $b_1,\dots,b_4$ are isomorphisms induced by the isomorphisms $\mathcal{F}^0 \cong (\mathcal{F}^1)^\vee\otimes\Omega^1_{\mathcal{C}/T}$, $\mathcal{F}^1 \cong (\mathcal{F}^0)^\vee\otimes\Omega^1_{\mathcal{C}/T}$ and Serre duality. So ξ becomes an isomorphism by five lemma. Hence ω is non-degenerate. □

Proposition 6.3. *For the 2-form constructed in Proposition 6.2, we have $d\omega = 0$.*

Proof. It suffices to show that the restriction of ω to any generic fiber $M^{\boldsymbol{\alpha}}_{\mathcal{C}_s}(\mathbf{t},\boldsymbol{\lambda})$ over $T\times\Lambda_r^{(n)}(d)$ is d-closed. We may assume $\boldsymbol{\lambda}$ generic, and so $\lambda^{(i)}_j - \lambda^{(i)}_{j'} \notin \mathbf{Z}$ for any i and $j\neq j'$. We also denote the

restriction by ω. Let $(\tilde{E},\tilde{\nabla},\{\tilde{l}^{(i)}_j\})$ be a universal family on $\mathcal{C}_s\times M^{\boldsymbol{\alpha}}_{\mathcal{C}_s}(\mathbf{t},\boldsymbol{\lambda})$ over $M^{\boldsymbol{\alpha}}_{\mathcal{C}_s}(\mathbf{t},\boldsymbol{\lambda})$. We set

$$\mathcal{F}^0 := \left\{ s\in \mathcal{E}nd(\tilde{E}) \,\middle|\, s|_{t_i\times M^{\boldsymbol{\alpha}}_{\mathcal{C}_s}(\mathbf{t},\boldsymbol{\lambda})}(\tilde{l}^{(i)}_j)\subset \tilde{l}^{(i)}_j \quad \text{for any } i,j \right\}$$
$$\mathcal{F}^1 := \left\{ s\in \mathcal{E}nd(\tilde{E})\otimes\Omega^1_{\mathcal{C}_s}(t_1+\cdots+t_n) \,\middle|\, \mathrm{res}_{t_i\times M^{\boldsymbol{\alpha}}_{\mathcal{C}_s}(\mathbf{t},\boldsymbol{\lambda})}(s)(\tilde{l}^{(i)}_j)\subset \tilde{l}^{(i)}_{j+1} \quad \text{for any } i,j \right\}$$
$$\nabla_{\mathcal{F}^\bullet}:\mathcal{F}^0\longrightarrow\mathcal{F}^1;\quad \nabla_{\mathcal{F}^\bullet}(s)=\tilde{\nabla}\circ s - s\circ\tilde{\nabla}.$$

We also put

$$\mathbf{V}:=\ker\tilde{\nabla}^{an}|_{(\mathcal{C}_s\setminus\{t_1,\dots,t_n\})\times M^{\boldsymbol{\alpha}}_{\mathcal{C}_s}(\mathbf{t},\boldsymbol{\lambda})}.$$

Since there is a canonical isomorphism

$$\ker\nabla^{an}_{\mathcal{F}^\bullet}|_{(\mathcal{C}_s\setminus\{t_1,\dots,t_n\})\times M^{\boldsymbol{\alpha}}_{\mathcal{C}_s}(\mathbf{t},\boldsymbol{\lambda})}\xrightarrow{\sim}\mathcal{E}nd(\mathbf{V}),$$

we obtain a canonical homomorphism

$$\ker\nabla^{an}_{\mathcal{F}^\bullet}|_{\mathcal{C}_s\times M^{\boldsymbol{\alpha}}_{\mathcal{C}_s}(\mathbf{t},\boldsymbol{\lambda})}\longrightarrow j_*(\mathcal{E}nd(\mathbf{V})),\tag{11}$$

where $j:(\mathcal{C}_s\setminus\{t_1,\dots,t_n\})\times M^{\boldsymbol{\alpha}}_{\mathcal{C}_s}(\mathbf{t},\boldsymbol{\lambda})\hookrightarrow\mathcal{C}_s\times M^{\boldsymbol{\alpha}}_{\mathcal{C}_s}(\mathbf{t},\boldsymbol{\lambda})$ is the canonical inclusion. We can check by local calculations that the homomorphism (11) is in fact an isomorphism. We can also check by local calculations that $\nabla^{an}_{\mathcal{F}^\bullet}:(\mathcal{F}^0)^{an}\to(\mathcal{F}^1)^{an}$ is surjective. So we obtain an isomorphism

$$\mathbf{R}^1(\pi_{M^{\boldsymbol{\alpha}}_{\mathcal{C}_s}(\mathbf{t},\boldsymbol{\lambda})})_*((\mathcal{F}^\bullet)^{an})\xrightarrow{\sim}R^1(\pi_{M^{\boldsymbol{\alpha}}_{\mathcal{C}_s}(\mathbf{t},\boldsymbol{\lambda})})_*(j_*(\mathcal{E}nd(\mathbf{V})))$$

of analytic sheaves. Then ω corresponds to a canonical pairing

$$R^1(\pi_{M^{\boldsymbol{\alpha}}_{\mathcal{C}_s}(\mathbf{t},\boldsymbol{\lambda})})_*(j_*(\mathcal{E}nd(\mathbf{V})))\times R^1(\pi_{M^{\boldsymbol{\alpha}}_{\mathcal{C}_s}(\mathbf{t},\boldsymbol{\lambda})})_*(j_*(\mathcal{E}nd(\mathbf{V})))\longrightarrow R^2(\pi_{M^{\boldsymbol{\alpha}}_{\mathcal{C}_s}(\mathbf{t},\boldsymbol{\lambda})})_*(\pi^{-1}_{M^{\boldsymbol{\alpha}}_{\mathcal{C}_s}(\mathbf{t},\boldsymbol{\lambda})}(\mathcal{O}_{M^{\boldsymbol{\alpha}}_{\mathcal{C}_s}(\mathbf{t},\boldsymbol{\lambda})}))$$
$$([\{c_{\alpha\beta}\}],[\{c'_{\alpha\beta}\}])\mapsto[\mathrm{Tr}(c_{\alpha\beta}\circ c'_{\beta\gamma})].$$

Note that $R^2(\pi_{M^{\boldsymbol{\alpha}}_{\mathcal{C}_s}(\mathbf{t},\boldsymbol{\lambda})})_*(\pi^{-1}_{M^{\boldsymbol{\alpha}}_{\mathcal{C}_s}(\mathbf{t},\boldsymbol{\lambda})}(\mathcal{O}_{M^{\boldsymbol{\alpha}}_{\mathcal{C}_s}(\mathbf{t},\boldsymbol{\lambda})}))\cong\mathcal{O}_{M^{\boldsymbol{\alpha}}_{\mathcal{C}_s}(\mathbf{t},\boldsymbol{\lambda})}$.

Take any small open subset $M\subset M^{\boldsymbol{\alpha}}_{\mathcal{C}_s}(\mathbf{t},\boldsymbol{\lambda})$ and open covering $\{U_\alpha\}$ of $\mathcal{C}_s$ such that $\sharp\{\alpha|t_i\in U_\alpha\}=1$ for any i and that $\sharp(U_\alpha\cap\{t_1,\dots,t_n\})\le 1$ for any α. If we replace U_α and M sufficiently smaller, there exist a sheaf E_α on U_α such that $E_\alpha|_{U_\alpha\cap U_\beta}\cong\mathbf{C}^{\oplus r}_{U_\alpha\cap U_\beta}$ for any $\beta\neq\alpha$ and an isomorphism $\phi_\alpha:\mathbf{V}|_{U_\alpha\times M}\xrightarrow{\sim}E_\alpha\otimes\mathcal{O}_M$. For each α,β, we put

$$\varphi_{\alpha\beta}:=\phi_\alpha\circ\phi_\beta^{-1}:E_\beta\otimes\mathcal{O}_M|_{(U_\alpha\cap U_\beta)\times M}\xrightarrow{\phi_\beta^{-1}}\mathbf{V}|_{(U_\alpha\cap U_\beta)\times M}\xrightarrow{\phi_\alpha}E_\alpha\otimes\mathcal{O}_M|_{(U_\alpha\cap U_\beta)\times M}.$$

So $\mathbf{V}|_{\mathcal{C}_s\times M^{\boldsymbol{\alpha}}_{\mathcal{C}_s}(\mathbf{t},\boldsymbol{\lambda})}$ is given by transition functions $\{\varphi_{\alpha\beta}\}$. Next we consider a vector field $v\in H^0(M,\Theta_M)$. v correspnds to a derivation $D_v:\mathcal{O}_M\to\mathcal{O}_M$ which naturally induces a morphism

$$D_v:\mathcal{H}om(E_\beta|_{U_\alpha\cap U_\beta},E_\alpha|_{U_\alpha\cap U_\beta})\otimes\mathcal{O}_M\longrightarrow\mathcal{H}om(E_\beta|_{U_\alpha\cap U_\beta},E_\alpha|_{U_\alpha\cap U_\beta})\otimes\mathcal{O}_M.$$

v also corresponds to a morphism $f_v:\operatorname{Spec}\mathcal{O}_M[\epsilon]\to M$, where $\epsilon^2=0$. Then $(1_{\mathcal{C}_s}\times f_v)^*(\mathbf{V})$ corresponds to a glueing data $\{(1_{\mathcal{C}_s}\times f_v)^*(\varphi_{\alpha\beta})\}$. We can see the following equality:

$$(1_{\mathcal{C}_s}\times f_v)^*(\varphi_{\alpha\beta})=\varphi_{\alpha\beta}+\epsilon D_v(\varphi_{\alpha\beta}):E_\beta|_{U_\alpha\cap U_\beta}\otimes\mathcal{O}_M[\epsilon]\xrightarrow{\sim}E_\alpha|_{U_\alpha\cap U_\beta}\otimes\mathcal{O}_M[\epsilon]$$
$$a+\epsilon b\mapsto\varphi_{\alpha\beta}(a)+\epsilon(\varphi_{\alpha\beta}(b)+D_v(\varphi_{\alpha\beta})(a)).$$

So the isomorphism $\Theta_M\cong R^1(\pi_M)_*(j_*(\mathcal{E}nd(\mathbf{V}))_M)$ is given by

$$T_M\ni v\mapsto[\{\phi_\alpha^{-1}\circ D_v(\varphi_{\alpha\beta})\circ\phi_\beta\}]\in R^1(\pi_M)_*(j_*(\mathcal{E}nd(\mathbf{V}_M))).$$

and

$$\omega(u,v)=[\{\mathrm{Tr}(D_u(\varphi_{\alpha\beta})\circ D_v(\varphi_{\beta\gamma})\circ\varphi_{\gamma\alpha})\}]\in R^2(\pi_M)_*(\pi_M^{-1}(\mathcal{O}_M))\cong\mathcal{O}_M.$$

Thus we have

$$
\begin{aligned}
d\omega(u,v,w) &= D_u(\omega(v,w)) + D_v(\omega(w,u)) + D_w(\omega(u,v)) + \omega(w,[u,v]) + \omega([u,w],v) + \omega(u,[v,w]) \\
&= \Big\{ \mathrm{Tr}\Big(D_u(D_v(\varphi_{\alpha\beta}) \circ D_w(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha}) + D_v(D_w(\varphi_{\alpha\beta}) \circ D_u(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha}) \\
&\quad + D_w(D_u(\varphi_{\alpha\beta}) \circ D_v(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha}) + D_w(\varphi_{\alpha\beta}) \circ (D_uD_v - D_vD_u)(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha} \\
&\quad + (D_uD_w - D_wD_u)(\varphi_{\alpha\beta}) \circ D_v(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha} + D_u(\varphi_{\alpha\beta}) \circ (D_vD_w - D_wD_v)(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha} \Big) \Big\} \\
&= \Big\{ \mathrm{Tr}\Big(D_uD_v(\varphi_{\alpha\beta}) \circ D_w(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha} + D_w(\varphi_{\alpha\beta}) \circ D_uD_v(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha} \\
&\quad + D_uD_w(\varphi_{\alpha\beta}) \circ D_v(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha} + D_v(\varphi_{\alpha\beta}) \circ D_uD_w(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha} \\
&\quad + D_vD_w(\varphi_{\alpha\beta}) \circ D_u(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha} + D_u(\varphi_{\alpha\beta}) \circ D_vD_w(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha} \\
&\quad + D_u(\varphi_{\alpha\beta}) \circ D_v(\varphi_{\beta\gamma}) \circ D_w(\varphi_{\gamma\alpha}) + D_v(\varphi_{\alpha\beta}) \circ D_w(\varphi_{\beta\gamma}) \circ D_u(\varphi_{\gamma\alpha}) \\
&\quad + D_w(\varphi_{\alpha\beta}) \circ D_u(\varphi_{\beta\gamma}) \circ D_v(\varphi_{\gamma\alpha}) \Big) \Big\}.
\end{aligned}
$$

On the other hand, applying $D_uD_vD_w$ to $\varphi_{\alpha\beta} \circ \varphi_{\beta\gamma} = \varphi_{\alpha\gamma}$, we obtain the equality

$$
\begin{aligned}
&- D_uD_vD_w(\varphi_{\alpha\beta})\varphi_{\beta\gamma} + D_uD_vD_w(\varphi_{\alpha\gamma}) - \varphi_{\alpha\beta}D_uD_vD_w(\varphi_{\beta\gamma}) \\
&= D_uD_v(\varphi_{\alpha\beta}) \circ D_w(\varphi_{\beta\gamma}) + D_w(\varphi_{\alpha\beta}) \circ D_uD_v(\varphi_{\beta\gamma}) \\
&\quad + D_uD_w(\varphi_{\alpha\beta}) \circ D_v(\varphi_{\beta\gamma}) + D_v(\varphi_{\alpha\beta}) \circ D_uD_w(\varphi_{\beta\gamma}) \\
&\quad + D_vD_w(\varphi_{\alpha\beta}) \circ D_u(\varphi_{\beta\gamma}) + D_u(\varphi_{\alpha\beta}) \circ D_vD_w(\varphi_{\beta\gamma})
\end{aligned}
$$

which implies

$$
\begin{aligned}
-d\{\mathrm{Tr}(D_uD_vD_w(\varphi_{\alpha\beta})\varphi_{\beta\alpha})\} = \Big\{ \mathrm{Tr}\Big(& D_uD_v(\varphi_{\alpha\beta}) \circ D_w(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha} + D_w(\varphi_{\alpha\beta}) \circ D_uD_v(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha} \\
& + D_uD_w(\varphi_{\alpha\beta}) \circ D_v(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha} + D_v(\varphi_{\alpha\beta}) \circ D_uD_w(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha} \\
& + D_vD_w(\varphi_{\alpha\beta}) \circ D_u(\varphi_{\beta\gamma}) \circ \varphi_{\gamma\alpha} + D_u(\varphi_{\alpha\beta}) \circ D_vD_w(\varphi_{\beta\gamma})) \circ \varphi_{\gamma\alpha} \Big) \Big\}.
\end{aligned}
$$

We can also check the equality

$$
\begin{aligned}
&\mathrm{Tr}\Big(D_u(\varphi_{\alpha\beta}) \circ D_v(\varphi_{\beta\gamma}) \circ D_w(\varphi_{\gamma\alpha}) + D_v(\varphi_{\alpha\beta}) \circ D_w(\varphi_{\beta\gamma}) \circ D_u(\varphi_{\gamma\alpha}) \\
&\qquad\qquad + D_w(\varphi_{\alpha\beta}) \circ D_u(\varphi_{\beta\gamma}) \circ D_v(\varphi_{\gamma\alpha}) \Big) \\
&= \mathrm{Tr}\Big(- D_u(\varphi_{\alpha\beta})D_v(\varphi_{\beta\alpha})D_w(\varphi_{\alpha\beta})\varphi_{\beta\alpha} + D_u(\varphi_{\alpha\gamma})D_v(\varphi_{\gamma\alpha})D_w(\varphi_{\alpha\gamma})\varphi_{\gamma\alpha} \\
&\qquad\qquad - D_u(\varphi_{\beta\gamma})D_v(\varphi_{\gamma\beta})D_w(\varphi_{\beta\gamma})\varphi_{\gamma\alpha} \Big)
\end{aligned}
$$

which means

$$
\begin{aligned}
&- d\{\mathrm{Tr}(D_u(\varphi_{\alpha\beta})D_v(\varphi_{\beta\alpha})D_w(\varphi_{\alpha\beta})\varphi_{\beta\alpha})\} \\
&= \Big\{ \mathrm{Tr}\Big(D_u(\varphi_{\alpha\beta}) \circ D_v(\varphi_{\beta\gamma}) \circ D_w(\varphi_{\gamma\alpha}) + D_v(\varphi_{\alpha\beta}) \circ D_w(\varphi_{\beta\gamma}) \circ D_u(\varphi_{\gamma\alpha}) \\
&\qquad\qquad + D_w(\varphi_{\alpha\beta}) \circ D_u(\varphi_{\beta\gamma}) \circ D_v(\varphi_{\gamma\alpha}) \Big) \Big\}.
\end{aligned}
$$

Thus we can see that $d\omega(u,v,w) = 0$. □

Remark 6.1. (1) Put $(C, \mathbf{t}) = (\mathcal{C}_x, \tilde{\mathbf{t}}_x)$ for a point $x \in T$ and take $\boldsymbol{\lambda} \in \Lambda_r^{(n)}(d)$. Let (L, ∇_L) be a pair of a line bundle L on C and a connection $\nabla_L : L \to L \otimes \Omega^1_C(D(\mathbf{t}))$ such that $\mathsf{res}_{t_i}(\nabla_L) = \sum_{j=0}^{r-1} \lambda_j^{(i)}$. Consider the moduli space $M_C^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ of $\boldsymbol{\alpha}$-stable parabolic connections with the determinant (L, ∇_L). Then we can see by the same proof as Proposition 6.2 that the restriction $\omega|_{M_C^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda},L)}$ of ω to $M_C^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ is also non-degenerate. So $M_C^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ also has a symplectic structure.
(2) By Proposition 7.1, we obtain a splitting $\Theta_{M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)/\Lambda_r^{(n)}(d)} \cong \pi^*\Theta_T \oplus \Theta_{M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)/T\times\Lambda_r^{(n)}(d)}$. With respect to this splitting, we can lift ω to $\tilde{\omega} \in H^0(\Omega^2_{M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)/\Lambda_r^{(n)}(d)})$ and we have $d\tilde{\omega} = 0$. This 2-form $\tilde{\omega}$ is nothing but the 2-form considered in [10] and [11]. Note that we have $\tilde{\omega} \cdot v = 0$ for $v \in D(\pi^*\Theta_T)$, where D is the homomorphism given in Proposition 7.1.

7. Isomonodromic deformation

Let $\tilde{T} \to T$ be a universal covering. Then $RP_r(\mathcal{C},\tilde{\mathbf{t}}) \times_T \tilde{T} \to \tilde{T}$ becomes a trivial fibration and we can consider the set of constant sections

$$\mathcal{F}^{\sharp}_R = \left\{ \sigma : \tilde{T} \to RP_r(\mathcal{C},\tilde{\mathbf{t}})^{\sharp} \times_T \tilde{T} \right\},$$

where

$$RP_r(\mathcal{C},\tilde{\mathbf{t}})^{\sharp} = \coprod_{(x,\mathbf{a})\in T\times\mathcal{A}_r^{(n)}} RP_r(\mathcal{C}_x,\tilde{\mathbf{t}}_x)^{\sharp}_{\mathbf{a}}.$$

The pull back $\tilde{\mathcal{F}}^{\sharp}_M = \{\mathbf{RH}^{-1}(\sigma)\}$ of this constant sections determines a foliation on $M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)^{\sharp} \times_T \tilde{T}$ where $M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)^{\sharp} = \mathbf{RH}^{-1}(RP_r(\mathcal{C},\tilde{\mathbf{t}})^{\sharp})$. This foliation corresponds to a subbundle determined by the splitting

$$D^{\sharp} : \tilde{\pi}^*(\Theta^{an}_{\tilde{T}}) \to \Theta^{an}_{M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)^{\sharp}\times_T\tilde{T}}$$

of the analytic tangent map

$$\Theta^{an}_{M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)^{\sharp}\times_T\tilde{T}} \longrightarrow \tilde{\pi}^*\Theta^{an}_{\tilde{T}} \to 0,$$

where $\tilde{\pi} : M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)^{\sharp} \times_T \tilde{T} \to \tilde{T}$ is the projection. We will show that this splitting $D^{\sharp}$ is in fact induced by a splitting

$$D : \pi^*(\Theta_T) \to \Theta_{M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)}$$

of the algebraic tangent bundle, where $\pi : M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d) \to T$ is the projection. This splitting is nothing but the differential equation determined by monodromy preserving deformation.

Usually, the equation of monodromy preserving deformation is given by the following way: Assume $C = \mathbf{P}^1$ and take a Zariski open set $U \subset M^{\boldsymbol{\alpha}}_{\mathbf{P}^1\times T/T}(\tilde{\mathbf{t}},r,0)$ such that a universal family on $\mathbf{P}^1 \times U$ is given by $(\mathcal{O}^{\oplus r}_{\mathbf{P}^1\times T}, \nabla, \{\tilde{l}^{(i)}_j\})$, where ∇ is given by a connection matrix

$$\sum_{i=1}^{n} \frac{A_i dz}{z-t_i} \quad (A_i = (a^{(i)}_{jk})_{1\le j,k\le r} \in M_r(\mathcal{O}_U)).$$

Then the differential equation of monodromy preserving deformation is given by

$$\frac{\partial A_i}{\partial t_j} = -\frac{[A_i,A_j]}{t_i-t_j} \quad (t_i \neq t_j), \qquad \frac{\partial A_i}{\partial t_i} = \sum_{k\neq i} \frac{[A_i,A_k]}{t_i-t_k}. \tag{12}$$

The equation (12) is called Schlesinger equation and it is needless to say that this equation plays a great role in explicit description of many differential equations arising from monodromy preserving deformation. However, we want to describe monodromy preserving deformation even on the locus contracted by **RH**. So the equation (12) itself is not enough for the explicit geometric description of monodromy preserving deformation, because (12) is defined only on a Zariski open set and is even not defined in higher genus case. In order to obtain the splitting D, we will go back to how the equation (12) arise as a monodromy preserving deformation. Take a local constant section $\sigma : T' \to RP_r(\mathcal{C},\tilde{\mathbf{t}})^{\sharp} \times_T T'$, where T' is an analytic open subset of T. Then σ corresponds to a local system $\mathbf{V}_\sigma$ on $\mathcal{C}_{T'} \setminus ((\tilde{t_1})_{T'} + \cdots + (\tilde{t_n})_{T'})$ after shrinking T'. The canonical flat connction on $\mathbf{V}_\sigma \otimes \mathcal{O}_{\mathcal{C}_{T'}\setminus((\tilde{t_1})_{T'}+\cdots+(\tilde{t_n})_{T'})}$ extends to an integrable logarithmic connection

$$\nabla^{\sigma} : \tilde{E}_{\mathbf{RH}^{-1}(\sigma(T'))} \longrightarrow \tilde{E}_{\mathbf{RH}^{-1}(\sigma(T'))} \otimes \Omega^1_{\mathcal{C}_{T'}}(\log((\tilde{t_1})_{T'} + \cdots + (\tilde{t_n})_{T'})) \tag{13}$$

whose induced relative connection is just $\tilde{\nabla}|_{\mathcal{C}\times_T\mathbf{RH}^{-1}(\sigma(T'))}$. The equation (12) comes from the integrability condition for ∇^{σ}. Taking the tangent direction of $\sigma(T')$, we obtain the splitting D. Precisely, we have the following proposition:

Proposition 7.1. *There exists an algebraic splitting*

$$D : \pi^*(\Theta_T) \to \Theta_{M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)}$$

of the tangent map $\Theta_{M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)} \to \pi^*(\Theta_T)$ *whose pull-back to* $M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)^{\sharp} \times_T \tilde{T}$ *coincides with* $D^{\sharp}$.

Proof. We will define the corresponding homomorphism $\Theta_T \to \pi_*\Theta_{M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)}$. Take any affine open set $U \subset T$ and a vector field $v \in H^0(U,\Theta_T)$. v corresponds to a morphism $\iota^v : \operatorname{Spec}\mathcal{O}_U[\epsilon] \to T$ with $\epsilon^2 = 0$ such that the composite $U \hookrightarrow \operatorname{Spec}\mathcal{O}_U[\epsilon] \to T$ is just the inclusion $U \hookrightarrow T$. We denote the restriction of the universal family to $\mathcal{C}\times_T \pi^{-1}(U)$ simply by $(\tilde{E},\tilde{\nabla},\{\tilde{l}^{(i)}_j\})$. Consider the fiber product $\mathcal{C}\times_T \operatorname{Spec}\mathcal{O}_{\pi^{-1}(U)}[\epsilon]$ with respect to the canonical projection $\mathcal{C}\to T$ and the compsite $\operatorname{Spec}\mathcal{O}_{\pi^{-1}(U)}[\epsilon] \to \operatorname{Spec}\mathcal{O}_U[\epsilon] \xrightarrow{\iota^v} T$. We denote the pull-back of $D(\tilde{\mathbf{t}})$ by the morphism $\mathcal{C}\times_T \operatorname{Spec}\mathcal{O}_{\pi^{-1}(U)}[\epsilon] \to \mathcal{C}$ simply by $D(\tilde{\mathbf{t}})_{\mathcal{O}_{\pi^{-1}(U)}[\epsilon]}$. We call $(\mathcal{E},\nabla^{\mathcal{E}},\{(l_{\mathcal{E}})^{(i)}_j\})$ a horizontal lift of $(\tilde{E},\tilde{\nabla},\{\tilde{l}^{(i)}_j\})$ if

(1) $\mathcal{E}$ is a vector bundle on $\mathcal{C}\times_T \operatorname{Spec}\mathcal{O}_{\pi^{-1}(U)}[\epsilon]$,

(2) $\mathcal{E}|_{\tilde{t}_i\times\operatorname{Spec}\mathcal{O}_{\pi^{-1}(U)}[\epsilon]} = (l_{\mathcal{E}})^{(i)}_0 \supset \cdots \supset (l_{\mathcal{E}})^{(i)}_r = 0$ is a filtration by subbundles for $i=1,\dots,n$,

(3) $\nabla^{\mathcal{E}} : \mathcal{E} \to \mathcal{E}\otimes\Omega^1_{\mathcal{C}\times_T\operatorname{Spec}\mathcal{O}_{\pi^{-1}(U)}[\epsilon]/\pi^{-1}(U)}\left(\log\left(D(\tilde{\mathbf{t}})_{\mathcal{O}_{\pi^{-1}(U)}[\epsilon]}\right)\right)$ is a connection satisfying

  (a) $\nabla^{\mathcal{E}}(F^{(i)}_j(\mathcal{E})) \subset F^{(i)}_j(\mathcal{E})\otimes\Omega^1_{\mathcal{C}\times_T\operatorname{Spec}\mathcal{O}_{\pi^{-1}(U)}[\epsilon]/\pi^{-1}(U)}\left(\log D(\tilde{\mathbf{t}})_{\mathcal{O}_{\pi^{-1}(U)}[\epsilon]}\right)$, where $F^{(i)}_j(\mathcal{E})$ is given by $F^{(i)}_j(\mathcal{E}) := \ker\left(\mathcal{E}\to\mathcal{E}|_{\tilde{t}_i\times_T\operatorname{Spec}\mathcal{O}_{\pi^{-1}(U)}[\epsilon]}/(l^{\mathcal{E}})^{(i)}_j\right)$,

  (b) The curvature $\nabla^{\mathcal{E}}\circ\nabla^{\mathcal{E}} : \mathcal{E}\to\mathcal{E}\otimes\Omega^2_{\mathcal{C}\times_T\operatorname{Spec}\mathcal{O}_{\pi^{-1}(U)}[\epsilon]/\pi^{-1}(U)}\left(\log\left(D(\tilde{\mathbf{t}})_{\mathcal{O}_{\pi^{-1}(U)}[\epsilon]}\right)\right)$ is zero,

  (c) $(\mathrm{res}_{\tilde{t}_i\times_T\operatorname{Spec}\mathcal{O}_{\pi^{-1}(U)}[\epsilon]}(\tilde{\nabla}^{\mathcal{E}}) - \tilde{\lambda}^{(i)}_j)((l^{\mathcal{E}})^{(i)}_j) \subset (l^{\mathcal{E}})^{(i)}_{j+1}$ for any i,j, where $\tilde{\nabla}^{\mathcal{E}}$ is the relative connection over $\operatorname{Spec}\mathcal{O}_{\pi^{-1}(U)}[\epsilon]$ induced by $\nabla^{\mathcal{E}}$ and

  (d) $(\mathcal{E},\tilde{\nabla}^{\mathcal{E}},\{(l^{\mathcal{E}})^{(i)}_j\})\otimes\mathcal{O}_{\pi^{-1}(U)}[\epsilon]/(\epsilon) \cong (\tilde{E},\tilde{\nabla},\{\tilde{l}^{(i)}_j\})$.

Here we define the sheaf $\Omega^1_{\mathcal{C}\times_T\operatorname{Spec}\mathcal{O}_{\pi^{-1}(U)}[\epsilon]/\pi^{-1}(U)}\left(\log\left(D(\tilde{\mathbf{t}})_{\mathcal{O}_{\pi^{-1}(U)}[\epsilon]}\right)\right)$ as the coherent subsheaf of $\Omega^1_{\mathcal{C}\times_T\operatorname{Spec}\mathcal{O}_{\pi^{-1}(U)}[\epsilon]/\pi^{-1}(U)}\left(D(\tilde{\mathbf{t}})_{\mathcal{O}_{\pi^{-1}(U)}[\epsilon]}\right)$ locally generated by $\tilde{g}^{-1}d\tilde{g}$ and $d\epsilon$ for a local defining equation $\tilde{g}$ of $D(\tilde{\mathbf{t}})_{\mathcal{O}_{\pi^{-1}(U)}[\epsilon]}$ and the sheaf $\Omega^2_{\mathcal{C}\times_T\operatorname{Spec}\mathcal{O}_{\pi^{-1}(U)}[\epsilon]/\pi^{-1}(U)}\left(\log\left(D(\tilde{\mathbf{t}})_{\mathcal{O}_{\pi^{-1}(U)}[\epsilon]}\right)\right)$ as the coherent subsheaf of $\Omega^2_{\mathcal{C}\times_T\operatorname{Spec}\mathcal{O}_{\pi^{-1}(U)}[\epsilon]/\pi^{-1}(U)}\left(D(\tilde{\mathbf{t}})_{\mathcal{O}_{\pi^{-1}(U)}[\epsilon]}\right)$ locally generated by $\tilde{g}^{-1}d\tilde{g}\wedge d\epsilon$. Assume that the parabolic connection $(\tilde{E},\tilde{\nabla},\{\tilde{l}^{(i)}_j\})$ is locally given in a small affine open subset W of $\mathcal{C}\times U$ by a connection matrix $Ag^{-1}dg$, where g is a local defining equation of $(\tilde{t}_i\times\pi^{-1}(U))\cap W$, $A\in M_r(\mathcal{O}_W)$, $A((\tilde{t}_i\times\pi^{-1}(U))\cap W)$ is an upper triangular matrix and the parabolic structure $\{\tilde{l}^{(i)}_j\}_W$ is given by $(\tilde{l}^{(i)}_j)_W = (\overbrace{*,*,\dots,*}^{r-j},0,\dots,0)$. Let $\tilde{W}$ be the affine open subscheme of $\mathcal{C}\times\operatorname{Spec}\mathcal{O}_{\pi^{-1}(U)}[\epsilon]$ whose underlying space is W and let $\tilde{g}$ be a lift of g which is a local defining equation of $(\tilde{t}_i\times\operatorname{Spec}\mathcal{O}_{\pi^{-1}(U)}[\epsilon])\cap\tilde{W}$. If we denote the composite $\mathcal{O}_{\tilde{W}} \xrightarrow{d} \Omega^1_{\tilde{W}/\pi^{-1}(U)} = \mathcal{O}_{\tilde{W}}d\tilde{g}\oplus\mathcal{O}_{\tilde{W}}d\epsilon \to \mathcal{O}_{\tilde{W}}d\epsilon$ by d_ϵ, we can take a lift $\tilde{A}\in M_r(\mathcal{O}_{\tilde{W}})$ of A such that $d_\epsilon(\tilde{A})=0$ and $\tilde{A}((\tilde{t}_i\times\operatorname{Spec}\mathcal{O}_{\pi^{-1}(U)}[\epsilon])\cap\tilde{W})$ is an upper-triangular matrix. Then the connection matrix $\tilde{A}\tilde{g}^{-1}d\tilde{g}$ gives a local horizontal lift of $(\tilde{E},\tilde{\nabla},\{\tilde{l}^{(i)}_j\})|_W$. We can check that a local horizontal lift is unique up to an isomorphism. We put

$$\mathcal{F}^0 := \left\{ u\in\mathcal{E}nd(\tilde{E}) \,\middle|\, u|_{\tilde{t}_i\times\pi^{-1}(U)}(\tilde{l}^{(i)}_j)\subset\tilde{l}^{(i)}_j \quad (1\le i\le n, 0\le j\le r)\right\}$$

$$\mathcal{F}^1 := \left\{ u\in\mathcal{E}nd(\tilde{E})\otimes\tilde{\Omega}^1 \,\middle|\, \begin{array}{l} u(F^{(i)}_j(\tilde{E}))\subset F^{(i)}_j(\tilde{E})\otimes\tilde{\Omega}^1 \text{ for any } i,j \text{ and the image of} \\ F^{(i)}_j(\tilde{E}) \xrightarrow{u} \tilde{E}\otimes\tilde{\Omega}^1 \xrightarrow{\mathrm{res}_{\tilde{t}_i}} \tilde{E}|_{\tilde{t}_i\times\pi^{-1}(U)} \text{ is contained in } \tilde{l}^{(i)}_{j+1} \text{ for any } i,j \end{array}\right\}$$

$$\mathcal{F}^2 := \left\{ u\in\mathcal{E}nd(\tilde{E})\otimes\tilde{\Omega}^2 \,\middle|\, u(F^{(i)}_j(\tilde{E}))\subset F^{(i)}_{j+1}(\tilde{E})\otimes\tilde{\Omega}^2 \text{ for any } i,j\right\},$$

where we put

$$\tilde{\Omega}^1 := \Omega^1_{\mathcal{C}\times_T\pi^{-1}(U)/\pi^{-1}(U)}(D(\tilde{\mathbf{t}}))\oplus\mathcal{O}_{\mathcal{C}\times_T\pi^{-1}(U)}d\epsilon$$
$$\tilde{\Omega}^2 := \Omega^1_{\mathcal{C}\times_T\pi^{-1}(U)/\pi^{-1}(U)}(D(\tilde{\mathbf{t}}))\wedge d\epsilon$$
$$F^{(i)}_j(\tilde{E}) := \ker\left(\tilde{E}\to\tilde{E}|_{\tilde{t}_i\times\pi^{-1}(U)}/\tilde{l}^{(i)}_j\right).$$

Consider the complex

$$\mathcal{F}^0 \xrightarrow{d^0} \mathcal{F}^1 \xrightarrow{d^1} \mathcal{F}^2$$

defined by $d^0(u) = \tilde{\nabla} \circ u - u \circ \tilde{\nabla} + ud\epsilon$ and $d^1(\omega + ad\epsilon) = d\epsilon \wedge \omega + (\tilde{\nabla} \circ a - a \circ \tilde{\nabla}) \wedge d\epsilon$ for $u \in \mathcal{F}^0$, $\omega \in \mathcal{E}nd(\tilde{E}) \otimes g^{-1}dg$ and $a \in \mathcal{F}^0$ such that $\omega + ad\epsilon \in \mathcal{F}^1$.

We can see that an obstruction class for the existence of a horizontal lift of $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_i\})$ is in $\mathbf{H}^2(\mathcal{F}^\bullet)$ and the set of horizontal lifts is isomorphic to $\mathbf{H}^1(\mathcal{F}^\bullet)$ if it is not empty. We can easily check that $\mathcal{F}^\bullet$ is in fact an exact complex. Thus we have $\mathbf{H}^2(\mathcal{F}^\bullet) = 0$, $\mathbf{H}^1(\mathcal{F}^\bullet) = 0$ and there is a unique horizontal lift $(\mathcal{E}, \nabla^{\mathcal{E}}, \{(l^{\mathcal{E}})^{(i)}_j\})$ of $(\tilde{E}, \tilde{\nabla}, \{\tilde{l}_i\})$. We can check that $v \mapsto (\mathcal{E}, \tilde{\nabla}^{\mathcal{E}}, \{(l^{\mathcal{E}})^{(i)}_j\})$ gives an $\mathcal{O}_T$-homomorphism $\Theta_T \to \pi_*(\Theta_{M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)})$. Thus we obtain an $\mathcal{O}_{M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)}$-homomorphism

$$D : \pi^*(\Theta_T) \to \Theta_{M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)}$$

which is a splitting of the surjection $\Theta_{M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)} \to \pi^*(\Theta_T)$. By construction we can see that the pull-back of D to $M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d)^\sharp \times \tilde{T}$ coincides with $D^\sharp$. □

Remark 7.1. (1) We can see by construction that the image $D(\pi^*\Theta_T)$ is in fact contained in the relative tangent bundle $\Theta_{M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)/\Lambda^{(n)}_r(d)}$.
(2) As is explained in section 2, the subbundle $D(\pi^*\Theta_T) \subset \Theta_{M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}},r,d)}$ satisfies an integrability condition and determies a foliation $\mathcal{F}_M$ on $M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d)$. By the proof of Proposition 7.1, we can see that $\{\mathbf{RH}(\mathcal{L}) | \mathcal{L} \in \mathcal{F}_M\}$ coincides with $\mathcal{F}_R$.

Proof of Theorem 2.3. Take any path $\gamma : [0, 1] \to T$ and a point $x \in M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d)$ such that $\pi(x) = \gamma(0)$. We take a lift $\tilde{x} \in M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d) \times_T \tilde{T}$. We can easily lift γ to a path $\tilde{\gamma} : [0, 1] \to RP_r(\mathcal{C}, \tilde{\mathbf{t}}) \times_{\mathcal{A}^{(n)}_r} \Lambda^{(n)}_r(d) \times_T \tilde{T}$ such that $\tilde{\gamma}[0, 1]$ is contained in a leaf of $\mathcal{F}_R \times_{\mathcal{A}^{(n)}_r} \Lambda^{(n)}_r(d)$ and $\mathbf{RH}(\tilde{x}) = \tilde{\gamma}(0)$. Since a leaf of $\tilde{\mathcal{F}}_M$ pathing through $\tilde{x}$ is locally isomorphic to $\tilde{T}$, we can take a unique local lift $\delta_\epsilon : [0, \epsilon) \to M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d) \times_T \tilde{T}$ of $\tilde{\gamma}$ for small $\epsilon > 0$ such that $\delta_\epsilon(0) = \tilde{x}$ and $\delta_\epsilon([0, \epsilon])$ is contained in a leaf of $\tilde{\mathcal{F}}_M$. Consider the set

$$A := \left\{ t \in (0, 1] \;\middle|\; \begin{array}{l} \text{there exists a lift } \delta_t : [0, t) \to M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d) \times_T \tilde{T} \text{ of } \tilde{\gamma}|_{[0,t)} \\ \text{such that } \delta(0) = \tilde{x} \text{ and } \delta([0, t)) \text{ is contained in a leaf of } \tilde{\mathcal{F}}_M \end{array} \right\}$$

and put $a := \sup A$. Assume that $a < 1$. Then there is a sequence $\{t_n\}_{n \geq 0}$ in A such that $\lim_{n\to\infty} t_n = a$. Since $\mathbf{RH} : M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d) \times_T \tilde{T} \to RP_r(\mathcal{C}, \tilde{\mathbf{t}}) \times_{\mathcal{A}^{(n)}_r} \Lambda^{(n)}_r(d) \times_T \tilde{T}$ is a proper morphism, $\mathbf{RH}^{-1}(\tilde{\gamma}[0, a])$ is compact. Thus the sequence $\{\delta_{t_n}(t_n)\}$ of $\mathbf{RH}^{-1}(\tilde{\gamma}[0, a])$ has a subsequence $\{\delta_{t_{n_k}}(t_{n_k})\}$ which is convergent in $\mathbf{RH}^{-1}(\tilde{\gamma}[0, a])$. We put $\tilde{x}_a := \lim_{k\to\infty} \delta_{t_{n_k}}(t_{n_k}) \in \mathbf{RH}^{-1}(\tilde{\gamma}[0, a])$. Then $\mathbf{RH}(\tilde{x}_a) = \tilde{\gamma}(a)$ and we can take a local lift $\tilde{\delta}_a : (a - \epsilon, a + \epsilon) \to M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d) \times_T \tilde{T}$ of $\tilde{\gamma}|_{(a-\epsilon,a+\epsilon)}$ for small $\epsilon > 0$ such that $\tilde{\delta}_a(a) = \tilde{x}_a$ and $\tilde{\delta}_a((a - \epsilon, a + \epsilon))$ is contained in a leaf of $\tilde{\mathcal{F}}_M$. Gluing $\tilde{\delta}_a$ and δ_{t_n} for some $t_n \in A$ with $t_n > a - \epsilon$, we can obtain a lift $\delta_a : [0, a + \epsilon) \to M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d) \times_T \tilde{T}$ of $\tilde{\gamma}|_{[0,a+\epsilon)}$ whose image is contained in a leaf of $\tilde{\mathcal{F}}_M$. Thus $a + \epsilon \in A$ which contradicts the choice of a. So we have $a = 1$ and we can obtain a lift $\delta_1 : [0, 1] \to M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d) \times_T \tilde{T}$ of $\tilde{\gamma}$ whose image is contained in a leaf of $\tilde{\mathcal{F}}_M$. Taking the image by the projection $M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d) \times_T \tilde{T} \to M^{\boldsymbol{\alpha}}_{\mathcal{C}/T}(\tilde{\mathbf{t}}, r, d)$, we obtain an $\mathcal{F}_M$-horizontal lift δ of γ such that $\delta(0) = x$. □

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